

OELJEKLAUS-TOMA MANIFOLDS AND ARITHMETIC INVARIANTS

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ABSTRACT. We consider Oeljeklaus-Toma manifolds coming from number fields with precisely one complex place. Our general theme is to relate the geometry with the arithmetic. We show that just knowing the fundamental group allows to recover the number field. We also show that this fails if there are more complex places. The first homology turns out to relate to an interesting ideal. We compute the volume in terms of the discriminant and regulator of the number field. Is there a conceptual reason for this? We explore this and see what happens if we (entirely experimentally!) regard them as “baby siblings” of hyperbolic manifolds coming from number fields. We ask the same questions and obtain similar answers, but ultimately it remains unclear whether we are chasing ghosts or not.

Unless stated otherwise, in this note the term “Oeljeklaus-Toma manifold” refers to the construction $X := X(K; \mathcal{O}_K^{\times,+})$ [OT05, §1] for an arbitrary number field K with $s \geq 1$ real places and *precisely one* complex place:

$$X := (\mathbf{H}^s \times \mathbf{C}) / (\mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}),$$

where \mathbf{H} refers to the complex upper half plane. Such a space X cannot carry a Kähler metric, but becomes a locally conformally Kähler (= LCK) manifold by a metric found in [OT05]. These generalize the Inoue surfaces of type S^0 [Ino74, §2]. LCK manifolds are an exciting generalization of Kähler manifolds and the Oeljeklaus-Toma manifolds supply a load of non-Kähler examples with rich additional properties [OV13]. But even more excitingly, there is already ample evidence that their complex geometry truly reflects arithmetical properties of the number field K , e.g. [PV12], [Vul14], [Dub14]. This note intends to show that this also applies to the volume and fundamental group.

For any number field K let K^g be its Galois closure over the rationals; for a group π write $\pi_{\text{ab}} := \pi/[\pi, \pi]$ for its maximal abelian quotient, π_{tor} for its torsion subgroup, and $\pi_{\text{fr}} := \pi/\pi_{\text{tor}}$ for its maximal torsion-free quotient.

Firstly, we can reconstruct the number field K just from the fundamental group of X by the following recipe:

Proposition 1. *Let X be an Oeljeklaus-Toma manifold, but it suffices to know $\pi := \pi_1(X, *)$. Then we have the short exact sequence*

$$(0.1) \quad 1 \longrightarrow \varkappa \longrightarrow \pi \longrightarrow \pi_{\text{ab,fr}} \longrightarrow 1,$$

where \varkappa is just defined as the kernel. There is a (non-canonical) isomorphism $\varkappa \simeq \mathbf{Z}^n$ for some n . Sequence 0.1 induces an action $\pi_{\text{ab,fr}} \curvearrowright \varkappa$ and therefore a representation

$$\rho : \pi_{\text{ab,fr}} \longrightarrow \text{GL}_n(\mathbf{Z}),$$

well-defined up to conjugation. Then K is uniquely determined and $K^g = \mathbf{Q}(\{\lambda\})$, where $\{\lambda\}$ is the set of complex eigenvalues of ρ .

Date: March 10, 2015.

The author has benefitted from the GK1821 “Cohomological Methods in Geometry”.

Quite differently, if we take the spaces $X(K; U)$ of [OT05] with $t > 1$ complex places into consideration, we will exhibit two diffeomorphic such manifolds whose underlying number fields are different, even after taking the Galois closure. So no reconstruction of K^g as above can be possible for $t > 1$. See Example 2 in the text.

Secondly, it was already shown in [OT05] that the first Betti number of X agrees with the number of real places of K . It turns out that the torsion in the first homology group carries also some rather subtle arithmetic information:

Proposition 2. *Let X be an Oeljeklaus-Toma manifold. Then there is a canonical isomorphism*

$$H_1(X, \mathbf{Z})_{\text{tor}} \xrightarrow{\sim} \mathcal{O}_K / J(\mathcal{O}_K^{\times,+}),$$

where $J(\mathcal{O}_K^{\times,+})$ is the ideal generated by all $1 - u$ with $u \in \mathcal{O}_K^{\times,+}$. In particular, $H_1(X, \mathbf{Z})_{\text{tor}}$ is generated by at most $s + 2$ elements.

See Prop. 6 in the text for a more general version. Although not spelled out explicitly in their article [PV12], this result follows directly from the ideas of Parton and Vuletescu. We give a number of examples where this torsion subgroup is quite large, notably

$$\#H_1(X, \mathbf{Z})_{\text{tor}} = 2^2 \cdot 5^2 \cdot 7 \cdot 967 \cdot 1649120827309715616889.$$

See Example 1 below. We also show that for any given $m \geq 1$ there exists an Inoue surface of type S^0 with $H_1(X, \mathbf{Z})_{\text{tor}} \cong \mathbf{Z}/m$, see Prop. 10. As far as I can tell, there is no easy general formula to compute the order of the torsion group. However, a computer can determine it rather quickly in any given case.

Thirdly, if we use the canonical metric on X , the volume of X relates – in a rather simple way – to the picture of Dirichlet’s analytic class number formula. This leads to a connection to the zeta function of the number field, namely:

Proposition 3. *Let K be a number field with $s \geq 1$ real places and one complex place and $X := X(K; \mathcal{O}_K^{\times,+})$ the associated Oeljeklaus-Toma manifold equipped with its canonical metric. Then*

$$(0.2) \quad \text{Vol}(X) = \frac{(s+1)}{4^s \cdot 2^{s^2}} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \cdot R_K$$

$$(0.3) \quad = \frac{(s+1)}{2^{3s+s^2} h_K} \cdot |\Delta_{K/\mathbf{Q}}| \cdot \pi^{-1} \cdot \text{res}_{s=1} \zeta_K(s),$$

where $\Delta_{K/\mathbf{Q}}$ denotes the discriminant of K , R_K is the Dirichlet regulator, h_K the class number of K , ζ_K the Dedekind zeta function of K .

The essence of this computation is Equation 0.2. The second line is just a reformulation in terms of the analytic class number formula. The volumes behave in a certain fashion, quite similar to analogous results in the theory of hyperbolic manifolds:

Proposition 4. *Among all Oeljeklaus-Toma manifolds X of some fixed dimension, there is a smallest possible volume, realized by at least one, but at most finitely many Oeljeklaus-Toma manifolds. There is a unique smallest Oeljeklaus-Toma surface, its volume is*

$$\text{Vol}(X) = 0.337146 \dots$$

It is the one coming from the number field

$$K := \mathbf{Q}[T]/(T^3 - T + 1).$$

With the assistance of the computer we can also provide the smallest possible Oeljeklaus-Toma manifolds in dimensions ≤ 6 . In all these cases the smallest manifold always comes from the number field with given s and $t = 1$ whose discriminant has the smallest absolute value. The reader should not hasten to believe that this might be a general fact. There are pairs with increasing discriminant, while the volume decreases. See §11.

We should also explain why one could expect the volume of an Oeljeklaus-Toma manifold to be related to the arithmetic of K at all – by parallels to a much more complicated, but also much older theory:

1. PARALLELS TO ARITHMETIC HYPERBOLIC MANIFOLDS

1.1. Phenomenology. We want to look at the geometry of Oeljeklaus-Toma manifolds by investigating to what extent there exist some parallels to the behaviour of hyperbolic manifolds in dimension ≥ 3 . In the following list it might not be clear (especially to the author of these lines) which analogies are meaningful, and which are accidents¹:

- (1) Going through the universal covering space, we have the presentation

$$X = \mathbf{H}_n / \Gamma \quad \text{versus} \quad X' = (\mathbf{H}^s \times \mathbf{C}) / \Gamma',$$

where X denotes a hyperbolic n -manifold; \mathbf{H}_n denotes hyperbolic n -space with the hyperbolic metric. On the right hand side X' is an Oeljeklaus-Toma manifold and $\mathbf{H}^s \times \mathbf{C}$ is equipped with the Oeljeklaus-Toma locally conformally Kähler metric. In either case, Γ is a discrete, finite covolume subgroup of the relevant isometry groups.

- (2) In dimension ≥ 3 finite volume hyperbolic manifolds are determined (up to isometry) by their fundamental groups (Mostow-Prasad Rigidity). For Oeljeklaus-Toma manifolds we find in Proposition 1 that they are also uniquely determined by their fundamental groups. Up to diffeomorphism this is just Mostow Rigidity for real solvmanifolds, but we even get the number field K back.
- (3) As a consequence of Rigidity, the volume of a hyperbolic n -manifold for $n \geq 3$ is a *topological* invariant. Similarly, once we pick an overall normalization for the Kähler potential on $\mathbf{H}^s \times \mathbf{C}$, diffeomorphic Oeljeklaus-Toma manifolds admit a canonical notion of volume.
- (4) Among the hyperbolic manifolds, there is the special class of ‘arithmetic’ ones. They come from arithmetic groups defined through number fields K , and their volume relates to the special value $\zeta_K(2)$ of the zeta function of the number field. For example, there is Humbert’s formula, discovered in 1919,

$$(1.1) \quad \text{Vol}(\mathbf{H}_3 / \text{PSL}_2(\mathcal{O}_K)) = \frac{1}{4} \cdot |\Delta_{K/\mathbf{Q}}|^{\frac{3}{2}} \cdot \pi^{-2} \cdot \zeta_K(2)$$

for K an imaginary quadratic number field. This can be generalized broadly, for example encompassing number fields with $s \geq 1$ real and $t = 1$ complex places. One switches to product-hyperbolic geometries,

$$\Gamma \subset \mathbf{H}^s \times \mathbf{H}_3^t, \quad X := (\mathbf{H}^s \times \mathbf{H}_3^t) / \Gamma$$

and still gets volume formulas of the shape

$$(1.2) \quad \text{Vol}(X) = (\text{rational factor}) \cdot |\Delta_{K/\mathbf{Q}}|^{\frac{3}{2}} \cdot \pi^{-s-2t} \cdot \zeta_K(2).$$

¹Especially since mathematics does not have accidents. It has, however, plenty of room for misleading analogies.

See [Bro13, Thm. 3.2] for a whole panorama of related volume computations. On the other hand, Proposition 3 shows that the volume of Oeljeklaus-Toma manifolds is

$$(\text{rational factor}) \cdot \pi^{-1} \cdot \text{res}_{s=1} \zeta_K(s).$$

This is certainly far less exciting than obtaining $\zeta_K(2)$, but it is pleasant that to see that the Oeljeklaus-Toma LCK metric, i.e. a metric with special complex-geometric properties, *entirely naturally* leads to this kind of volume formula. In some sense, this behaviour was already built-in within Tricerri's LCK metric for the S^0 Inoue surface.

- (5) Finally, the work of Thurston and Jorgensen [Gro81] has shown a very interesting structure on the volume distribution among hyperbolic 3-manifolds. In particular, there is a unique hyperbolic 3-manifold of smallest volume. Asking the same question for Oeljeklaus-Toma manifolds, one also finds that there is (in each dimension) a smallest volume, attained by at least one, and at most finitely many Oeljeklaus-Toma manifolds. Incidentally, the smallest hyperbolic 3-manifold (the Weeks manifold) is arithmetic, and comes from the *same* number field as the smallest Oeljeklaus-Toma manifold. Quite possibly, however, this is just a sporadic effect of small numbers.

hyperbolic 3-manifold	Oeljeklaus-Toma
universal cover \mathbf{H}_3	universal cover $\mathbf{H} \times \mathbf{C}$
$\text{SL}_2(\mathcal{O}_K)$	$\supset \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}$
hyperbolic metric	locally conformally Kähler
$\zeta_K(2)$	residue at $\zeta_K(1)$,

There is one thing I should say very clearly: The above analogies might all be purely phenomenological. Many of them could be attributed to Oeljeklaus-Toma manifolds being real solvmanifolds. However, it remains curious that the LCK metric (i.e. a metric chosen for its special holomorphic features) leads to a zeta value volume so naturally.

Remark 1. On the other hand, it must be said that all of the above analogies completely collapse if one considers number fields with $t > 1$ complex places as well:

- (1) In this case the formation of $X(K; U)$ depends on a choice, which is perhaps unnatural.
- (2) It seems to be a very delicate question whether there exists some U so that $X(K; U)$ can be equipped with a LCK metric. See Remark 2 for recent work on this issue. This makes it difficult to speak of volume at all.
- (3) We show in Example 2 that a matching of fundamental groups, although it still implies being diffeomorphic, does not allow to reconstruct the field K . So even if one can come up with a normalized LCK metric of some sort, like the one of Battisti [Dub14, Appendix], it seems improbable that there is a unique choice within each diffeomorphism class.

2. PREPARATIONS

We shall exclusively use Poincaré's upper half plane model for the hyperbolic 2-space \mathbf{H} . The Iwasawa decomposition of the group $\text{SL}_2(\mathbf{R})$ is the homeomorphism $\text{SL}_2(\mathbf{R}) \approx$

$K \times A \times N$, where

$$K := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad A := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \quad N := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$

(for $a > 0$, $b \in \mathbf{R}$) are three subgroups; $K = \mathrm{SO}_2(\mathbf{R})$ is a maximal compact subgroup. We can use this to obtain a very pleasant parametrization of the complex upper half plane $\mathbf{H} := \{x + iy \mid y > 0\} \subset \mathbf{C}$, namely

$$A \cdot N = \frac{\mathrm{SL}_2(\mathbf{R})}{K} = \frac{\mathrm{SL}_2(\mathbf{R})}{\mathrm{SO}_2(\mathbf{R})} = \mathbf{H}.$$

Let us recall the details: $\mathrm{SL}_2(\mathbf{R})$ acts on \mathbf{H} via the standard Möbius action, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}$$

for $z \in \mathbf{H}$ a complex number. Since

$$A \cdot N = \left\{ \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mid a \in \mathbf{R}_{>0}^\times \right\},$$

the orbit of $i \in \mathbf{H}$ under the action of $A \cdot N$ unwinds as

$$(2.1) \quad A \cdot N \cong \mathbf{H} \quad \left(\sqrt{y} \begin{pmatrix} x \\ \frac{1}{\sqrt{y}} \end{pmatrix} \right) \cdot i = x + iy \quad \in \mathbf{H}$$

for $x \in \mathbf{R}$ and $y > 0$; this is obviously simply transitive.

3. OELJEKLAUS-TOMA MANIFOLDS

Let K be a number field with s real places and t complex places. Let $n := [K : \mathbf{Q}] = s + 2t$; the ring of integers \mathcal{O}_K is a free \mathbf{Z} -module of rank n and the group of units \mathcal{O}_K^\times decomposes as $\mathcal{O}_{K,\mathrm{tor}}^\times \times \mathbf{Z}^{s+t-1}$, where the torsion subgroup $\mathcal{O}_{K,\mathrm{tor}}^\times$ is the group of roots of unity in K . Whenever $s \geq 1$, we necessarily have $\mathcal{O}_{K,\mathrm{tor}}^\times = \{\pm 1\}$. Suppose $\sigma_1, \dots, \sigma_s : K \rightarrow \mathbf{R}$ are the real embeddings and $\sigma_{s+1}, \dots, \sigma_{s+t}, \overline{\sigma_{s+1}}, \dots, \overline{\sigma_{s+t}} : K \rightarrow \mathbf{C}$ the t complex conjugate pairs of complex embeddings (the numbering and choice of complex conjugate partners is non-canonical, but does not affect any of the following). Write $\mathcal{O}_K^{\times,+} := \{x \in \mathcal{O}_K^\times \mid \sigma_j(x) > 0 \text{ for } 1 \leq j \leq s\}$ for the group of totally positive units.

Suppose $s \geq 1$ and $t \geq 1$. Following Oeljeklaus and Toma [OT05] we define

$$(3.1) \quad X(K; U) := \frac{\mathbf{H}^s \times \mathbf{C}^t}{\mathcal{O}_K \rtimes U},$$

where $U \subseteq \mathcal{O}_K^{\times,+}$ is a suitably chosen subgroup; it has to be “admissible” in the sense of [OT05]. The semi-direct product $\mathcal{O}_K \rtimes U$ is formed by letting U act on \mathcal{O}_K by multiplication. See [OT05, §1] for details. It is shown that the action is properly discontinuous, full rank, and holomorphic. As a result, $X(K; U)$ canonically becomes a compact complex manifold. In the present text we will mostly deal with the case $t = 1$. It is special in two ways: Firstly, there is a canonical choice for U because $U := \mathcal{O}_K^{\times,+}$ is always an admissible subgroup. Secondly, these $X(K; U)$ admit an LCK metric. We will review this in §6.

Remark 2. In fact by the work of Vuletescu [Vul14] and Battisti [Dub14, Appendix, Theorem 8], we now know that $X(K; U)$ as in Equation 3.1 admits an LCK metric if and only if for all $\alpha \in U$ one has $|\sigma_{s+1}(\alpha)| = \dots = |\sigma_{s+t}(\alpha)|$. For the case we are mostly interested in, i.e. $t = 1$, this condition is trivially met. See Dubickas [Dub14] for an extensive study whether this condition can be satisfied for $t > 1$.

Lemma 1. *On the \mathbf{H} -factors in $\mathbf{H}^s \times \mathbf{C}$ the Möbius action of the subgroup*

under the embedding

agrees with the Oeljeklaus-Toma action of the subgroup $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$. In particular, the subgroup in Equation 3.2 is isomorphic to the semi-direct product $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$.

$$\begin{aligned} \varphi : \mathcal{O}_K \rtimes (\mathcal{O}_K^\times, +)^2 &\longrightarrow \mathrm{SL}_2(\mathcal{O}_K) \\ (u, v) &\longmapsto \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ & \frac{1}{\sqrt{v}} \end{pmatrix}. \end{aligned}$$
$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto (ab, a^2) \quad \in \mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2.$$
$$x + iy = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i,$$
$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \cdot z = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i = (x + b) + iy$$
$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \cdot z = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i = a^2(x + iy)$$

4. THE COMMUTATOR SUBGROUP $[\pi, \pi]$

Next, we want to understand the structure of the commutator subgroup of $\mathcal{O}_K \rtimes U$. Since it will not be any more difficult to treat the general case, let us for the moment allow for U any subgroup of \mathcal{O}_K^\times . The following definition is suggested by the observations in the paper of Parton and Vuletescu [PV12, Proof of Thm. 4.2]:

Definition 1. For a number field K and an arbitrary subgroup $U \subseteq \mathcal{O}_K^\times$ define

$$J(U) := \{\text{ideal generated by } 1 - v \text{ for all } v \in U\} \subseteq \mathcal{O}_K.$$

Of course this ideal might just be the entire ring of integers. If one actually wants to compute this ideal, it will be convenient to reduce its definition to a finite number of generators:

Lemma 2. If $\varepsilon_1, \dots, \varepsilon_s$ are generators of the group U , then the ideal $J(U)$ can also be described as

$$J(U) = (1 - \varepsilon_1, \dots, 1 - \varepsilon_s).$$

Proof. It is clear that the $\{1 - \varepsilon_i\}_{i=1, \dots, s}$ generate a sub-ideal of $J(U)$, so it suffices to prove the converse inclusion. Observe that if $v, \tilde{v} \in U$ then

$$(4.1) \quad v(1 - \tilde{v}) + (1 - v) = 1 - v\tilde{v},$$

where the left-hand side lies in the ideal generated by $1 - v$ and $1 - \tilde{v}$. Therefore, for all products $v = \varepsilon_1^{n_1} \cdots \varepsilon_s^{n_s}$ with $n_1, \dots, n_s \in \mathbf{Z}$ we may inductively rewrite $1 - v$ along Equation 4.1 as an element in the ideal generated by the $\{1 - \varepsilon_i, 1 - \varepsilon_i^{-1}\}_{i=1, \dots, s}$. Here $1 - \varepsilon_i^{-1}$ occurs since Equation 4.1 just allows to reduce products, but not inverses. However, we also have

$$1 - v^{-1} = -v^{-1}(1 - v)$$

for all units $v \in U$, showing that the generators $1 - \varepsilon_i^{-1}$ for $i = 1, \dots, s$ are actually not needed. \square

Lemma 3. For a number field K and subgroups $U, V \subseteq \mathcal{O}_K^\times$ we have

$$J(U) + J(V) = J(U \cdot V),$$

where we have a sum of ideals on the left-hand side and $U \cdot V$ denotes the smallest subgroup of \mathcal{O}_K^\times containing both U and V .

Proof. Every element of $U \cdot V$ has the shape uv with $u \in U, v \in V$ since \mathcal{O}_K^\times is abelian. By Equation 4.1 the element $1 - uv$ lies in the ideal sum $J(U) + J(V)$. Conversely, any element of the sum can be written as $\sum a_i(1 - u_i)$ with $a_i \in \mathcal{O}_K$ and u_i (for each i) either in U or in V , so in either case $u_i \in U \cdot V$. \square

Proposition 5. Let K be a number field and $U \subseteq \mathcal{O}_K^\times$ a subgroup. Then for the semi-direct product $\pi := \mathcal{O}_K \rtimes U$ we have

$$[\pi, \pi] = \{(u, 1) \in \pi \mid u \in J(U)\}.$$

Analogously, in the subgroup of $\text{SL}_2(\mathcal{O}_K)$ defined in Equation 3.2 the commutator subgroup consists of all matrices

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$$

with $b \in J(U^2)$.

Proof. We have $(u, v)(\tilde{u}, \tilde{v}) = (u + v\tilde{u}, v\tilde{v})$ and $(u, v)^{-1} = (-uv^{-1}, v^{-1})$. Since the commutator subgroup $[\pi, \pi]$ is generated by commutators, it is actually generated by all elements of the shape

$$(4.2) \quad \begin{aligned} (u, v)(\tilde{u}, \tilde{v})(u, v)^{-1}(\tilde{u}, \tilde{v})^{-1} &= (u + v\tilde{u}, v\tilde{v})(-uv^{-1} - \tilde{u}v^{-1}\tilde{v}^{-1}, v^{-1}\tilde{v}^{-1}) \\ &= ((1 - \tilde{v})u - (1 - v)\tilde{u}, 1). \end{aligned}$$

This shows that the commutator subgroup $[\pi, \pi]$ is contained in the abelian group $(\mathcal{O}_K, 1)$ and then necessarily a subgroup. Furthermore, since $u, \tilde{u} \in \mathcal{O}_K$ are arbitrary, $[\pi, \pi]$ contains the subgroup $(I, 1)$ for $I = (1 - \tilde{v}, 1 - v)$, i.e. the ideal in \mathcal{O}_K generated by the elements $1 - \tilde{v}, 1 - v$. Since the latter is true for all $v \in U$ and $[\pi, \pi]$ is closed under addition in $(\mathcal{O}_K, 1)$, it follows that $[\pi, \pi]$ contains all elements $(x, 1)$ with $x \in J(U)$. Conversely, from Equation 4.2 it is clear that all elements in $[\pi, \pi]$ are of the shape $(x, 1)$ with $x \in J(U)$, proving the claim. The claim about SL_2 follows directly from Lemma 1. \square

Proposition 6. *Let K be a number field and $U \subseteq \mathcal{O}_K^\times$ a torsion-free subgroup $\neq 1$. Then for $\pi := \mathcal{O}_K \rtimes U$ the kernel \varkappa in the short exact sequence*

$$(4.3) \quad 1 \longrightarrow \varkappa \longrightarrow \pi \longrightarrow \pi_{\mathrm{ab}, \mathrm{fr}} \longrightarrow 1$$

is just the subgroup \mathcal{O}_K . Moreover, if U is admissible in the sense of [OT05], we have a canonical short exact sequence

$$(4.4) \quad 0 \longrightarrow \mathcal{O}_K/J(U) \longrightarrow H_1(X(K; U), \mathbf{Z}) \longrightarrow U \longrightarrow 0,$$

where $\mathcal{O}_K/J(U)$ is precisely the torsion subgroup. In particular, this group needs at most $s + 2t$ generators.

Proof. (after Parton and Vuletescu [PV12, Thm. 4.2]) We have the commutative diagram with exact rows

$$(4.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & [\pi, \pi] & \longrightarrow & \pi & \longrightarrow & \pi_{\mathrm{ab}} \longrightarrow 1 \\ & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \\ 1 & \longrightarrow & \mathcal{O}_K & \longrightarrow & \pi & \longrightarrow & U \longrightarrow 1, \end{array}$$

where the upper row is formed from abelianization and the lower row from the semi-direct product structure of π . The existence of the downward arrows follows from Prop. 5, along with $[\pi, \pi] = J(U)$, where $J(U)$ is an ideal in \mathcal{O}_K . Since we assume $U \neq 1$, $J(U)$ is not the zero ideal and therefore must have finite index in \mathcal{O}_K as an abelian group; this means that

$$m\mathcal{O}_K \subseteq [\pi, \pi] \subseteq \mathcal{O}_K$$

for some m . It is now an easy diagram chase² to see that $\ker(\pi_{\mathrm{ab}} \rightarrow U)$ is a pure torsion group, in fact it is annihilated by m . On the other hand since U is torsion-free, the kernel of $\pi_{\mathrm{ab}} \rightarrow U$ must contain the full torsion subgroup. We conclude that $\ker(\pi_{\mathrm{ab}} \rightarrow U)$ actually agrees with the torsion subgroup in π_{ab} , and via the snake map with $\mathcal{O}_K/J(U)$. Since the right-hand side downward arrow in Diagram 4.5 is moreover surjective, we deduce that $U = \pi_{\mathrm{ab}, \mathrm{fr}}$ and therefore \varkappa in Equation 4.3 agrees with \mathcal{O}_K . Finally, if U is admissible, we can form $X(K; U)$ and by the Hurewicz Theorem there is a canonical isomorphism

$$\pi_1(X(K; U), *)_{\mathrm{ab}} \xrightarrow{\sim} H_1(X(K; U), \mathbf{Z})$$

and our previous argument decomposes the left-hand side just in the shape of Equation 4.4. Since $\mathcal{O}_K \simeq \mathbf{Z}^{s+2t}$, every quotient group requires at most $s + 2t$ generators itself. \square

We should give some examples regarding the structure of $J(U)$. Firstly, it is easy to produce examples where the ideal is non-trivial:

²The Snake Lemma is false for arbitrary non-abelian groups, but it *does* hold for the specific Diagram 4.5. The essential reason is that all kernels and cokernels in this diagram exist. This would not necessarily hold for a general diagram of non-abelian groups.

Example 1. We give a few examples for the group of totally positive units, i.e. we consider the ideal norms $N(J(\mathcal{O}_K^{\times,+})) = \#\mathcal{O}_K/J(\mathcal{O}_K^{\times,+})$. We perform this computation for the number fields

$$F_m = \mathbf{Q}[T]/(T^3 - T + m) \quad G_m = \mathbf{Q}[T]/(T^7 - T - m) \quad H_m = \mathbf{Q}[T]/(T^3 - 2T - m),$$

the result spelled out in the respective column:

m	F_m	G_m	H_m
1	1	1	—
2	2^2	2^2	2
3	3^2	1	$2 \cdot 3^2$
4	2^3	2^2	—
5	19	1	2^4
6	—	2^2	$2 \cdot 3 \cdot 5 \cdot 11$
7	17	1	$2 \cdot 7 \cdot 109$

This table was generated by computer, see Code 2 on page 31 for details. The dashes “—” indicate whenever the given polynomial is not irreducible. Solely for the entertainment of the reader, let us also list a large example: For the randomly chosen number field $\mathbf{Q}[T]/(T^3 + 2T + 2000)$ one gets

$$\#\mathcal{O}_K/J(\mathcal{O}_K^{\times,+}) = 2^2 \cdot 5^2 \cdot 7 \cdot 967 \cdot 1649120827309715616889.$$

As far as I can tell, there does not seem to be an obvious pattern governing the structure of the ideal $J(U)$.

Remark 3. If the reader wants to produce infinite families of number fields so that $J(\mathcal{O}_K^{\times,+}) = (1)$ for all of them, the easiest way is to pick some number field L with $J(\mathcal{O}_L^{\times,+}) = (1)$. Then take any family L_i of number fields and consider the composita $K_i := L_i \cdot L$. Then we have

$$\mathcal{O}_{K_i} = J_{L_i}(\mathcal{O}_{L_i}^{\times,+})\mathcal{O}_{K_i} \subseteq J_{K_i}(\mathcal{O}_{L_i}^{\times,+}) \subseteq J_{K_i}(\mathcal{O}_{K_i}^{\times,+}),$$

where J_F refers to forming the ideal $J(U)$ with respect to the number field F .

Of course Proposition 6 provokes the question: What abelian groups can occur for $\mathcal{O}_K/J(U)$ at all? For example, for $s = t = 1$ we see that only finite abelian groups with at most three generators are possible; this is of course already in Inoue’s original paper [Ino74, §2, p. 274]. Do all of them really occur? I supply a crude ‘first approach’ for cyclic groups in Prop. 10 below, but it is not quite satisfactory.

There is also a completely different way to characterize \mathcal{O}_K inside $\mathcal{O}_K \rtimes U$ and it could serve as an alternative definition of \varkappa in the formulation of Proposition 1:

Proposition 7. *Let K be a number field and $U \subseteq \mathcal{O}_K^\times$ a subgroup. Consider the family of all subgroups*

$$\mathcal{H} := \{H \subseteq \mathcal{O}_K \rtimes U \mid H \simeq \mathbf{Z}^n \text{ for some } n \geq 0\}.$$

Then there is a maximal n which can occur, and all those realizing the maximal n are partially ordered by inclusion and there is a unique maximal H among them. In fact, this maximal H is the subgroup \mathcal{O}_K .

Proof. Suppose some non-trivial $H \in \mathcal{H}$ exists and let $(u, v) \in H$ be some element which is not the identity. Since H is abelian, all $(\tilde{u}, \tilde{v}) \in H$ must commute with (u, v) . By Equation 4.2 this forces

$$\mathbf{1}_H = (u, v)(\tilde{u}, \tilde{v})(u, v)^{-1}(\tilde{u}, \tilde{v})^{-1} = ((1 - \tilde{v})u - (1 - v)\tilde{u}, 1),$$

so

$$(4.6) \quad (1 - \tilde{v})u - (1 - v)\tilde{u} = 0.$$

Now we need a case distinction:

(1) Suppose $v \neq 1$. Then in the field K we can solve for \tilde{u} and find

$$(4.7) \quad \tilde{u} = \frac{1 - \tilde{v}}{1 - v}u.$$

For any given $\tilde{v} \in U$ it can be true or false that the right-hand side lies in \mathcal{O}_K (recall that u, v are fixed). We obtain that the largest subset of $\mathcal{O}_K \rtimes U$ of elements commuting with (u, v) is

$$(4.8) \quad C_{u,v} := \left\{ (\tilde{u}, \tilde{v}) \text{ so that } \tilde{u} = \frac{1 - \tilde{v}}{1 - v}u \in \mathcal{O}_K \right\}$$

and by definition as a centralizer this is actually a subgroup. The latter can also be checked directly. As H is abelian and contains (u, v) , we must have $H \subseteq C_{u,v}$. We compose the inclusion of H with the projection of the semi-direct product, i.e.

$$\begin{aligned} H \hookrightarrow C_{u,v} \hookrightarrow \mathcal{O}_K \rtimes U &\longrightarrow U \hookrightarrow \mathcal{O}_K^\times \simeq \mu_K \times \mathbf{Z}^{s+t-1} \\ (\tilde{u}, \tilde{v}) &\longmapsto \tilde{v}. \end{aligned}$$

Since Equation 4.7 implies that \tilde{u} can be computed from \tilde{v} (and u, v were fixed), this composition is actually injective. It follows that the \mathbf{Z} -rank of H can be at most $s + t - 1$.

(2) Suppose $v = 1$. Then Equation 4.6 becomes $(1 - \tilde{v})u = 0$. We have $u \neq 0$ since $(u, v) = (0, 1)$ would then be the identity element, which we had excluded. Hence, the elements in $\mathcal{O}_K \rtimes U$ commuting with (u, v) are precisely those with $\tilde{v} = 1$; and these are precisely those forming the subgroup $\mathcal{O}_K \simeq \mathbf{Z}^{s+2t}$. We always have $s + 2t > s + t - 1$, so we conclude that the subgroups of Equation 4.8 are never realizing the maximal \mathbf{Z} -rank. Instead, it follows that only those $H \subseteq \mathcal{O}_K$ with $H \simeq \mathbf{Z}^{s+2t}$ realize the maximal \mathbf{Z} -rank and among all such H contained in \mathcal{O}_K clearly the full group $\mathcal{O}_K \simeq \mathbf{Z}^{s+2t}$ is the unique maximal. \square

5. PROOF OF PROPOSITION 1

This is inspired from the perspective taken in the proof of [PV12, Thm. 4.2] by Parton and Vuletescu.

Proof of Prop. 1. Since X is an Oeljeklaus-Toma manifold, it is connected and therefore $\pi := \pi_1(X, x)$ is well-defined up to the inner automorphism coming from the choice of picking a base point. We denote by π_{ab} its maximal abelian quotient, and by $\pi_{\text{ab}, \text{fr}}$ the maximal torsion-free quotient of the latter. We may then define \varkappa just as the corresponding kernel in the short exact sequence

$$(5.1) \quad 1 \longrightarrow \varkappa \longrightarrow \pi \longrightarrow \pi_{\text{ab}, \text{fr}} \longrightarrow 1.$$

Since $\mathbf{H}^s \times \mathbf{C}$ is contractible and the action of $\mathcal{O}_K \rtimes \mathcal{O}_K^{\times, +}$ is easily checked to be free, the Oeljeklaus-Toma manifold $X = (\mathbf{H}^s \times \mathbf{C})/(\mathcal{O}_K \rtimes \mathcal{O}_K^{\times, +})$ is actually a classifying space for the group, i.e. its homotopy type is a $B(\mathcal{O}_K \rtimes \mathcal{O}_K^{\times, +}) = K(\mathcal{O}_K \rtimes \mathcal{O}_K^{\times, +}, 1)$, an Eilenberg-MacLane space: $\pi_1(X, x) \simeq \mathcal{O}_K \rtimes \mathcal{O}_K^{\times, +}$ (and moreover $\pi_i(X, x) = 0$ for $i \geq 2$, but we do not need this). By Prop. 6 we already know that for the group $\pi = \mathcal{O}_K \rtimes \mathcal{O}_K^{\times, +}$ our Equation 5.1 becomes

$$1 \longrightarrow \mathcal{O}_K \longrightarrow \pi \longrightarrow \mathcal{O}_K^{\times, +} \longrightarrow 1,$$

where K is our still unknown number field. Of course we just know \mathcal{O}_K as an abelian group under addition here; we do not know the ring structure. However, by the rank of rings of integers and Dirichlet's Unit Theorem, we know that there exist non-unique isomorphisms

$$(5.2) \quad \mathcal{O}_K \simeq \mathbf{Z}^{s+2t} \quad \text{and} \quad \mathcal{O}_K^{\times,+} \simeq \mathbf{Z}^{s+t-1},$$

where s, t are the number of real and complex places of K . We have $t = 1$ by assumption³. We recall that for any short exact sequence of groups as in Equation 5.1 there is a morphism $\rho : \pi_{\text{ab}, \text{fr}} \rightarrow \text{Out}(\mathcal{K})$, $\phi \mapsto (x \mapsto \tilde{\phi} x \tilde{\phi}^{-1})$, where $\tilde{\phi}$ is any lift of $\phi \in \pi_{\text{ab}, \text{fr}}$ to π and “Out” denotes the group of outer automorphisms, i.e. automorphisms modulo conjugations. Since \mathcal{O}_K is abelian and conjugations are trivial, we may lift this ρ to take values in $\text{Aut}(\mathcal{K})$, and moreover ρ recovers the action used in the formation of the semi-direct product. From Equation 5.2 we already know that ρ becomes

$$\rho : \mathcal{O}_K^{\times,+} \longrightarrow \text{GL}(\mathbf{Z}^{s+2})$$

in our particular situation. Choosing a different splitting would just change ρ into an equivalent representation. Since we know that $\pi = \mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}$ was formed by letting $\mathcal{O}_K^{\times,+}$ act by multiplication on \mathcal{O}_K , we know that ρ must (up to conjugation) be precisely the multiplication action. Hence, the minimal polynomial of any $\alpha \in \mathcal{O}_K^{\times,+}$ acting on \mathbf{Z}^{s+2} will be nothing but its minimal polynomial as an algebraic number. A generically chosen $\alpha \in \mathcal{O}_K^{\times,+}$ is a primitive element for the field extension K/\mathbf{Q} , so K is uniquely determined. Note moreover that any conjugation does not affect minimal polynomials, so it does not matter that we only know ρ up to equivalence. We conclude that all complex eigenvalues of all $\alpha \in \mathcal{O}_K^{\times,+}$ lie in $K^{\mathfrak{g}}$. Now we are done if adjoining all of them to \mathbf{Q} contains K (because adjoining all roots of the minimal polynomials produces a Galois extension and the smallest Galois extension containing K is $K^{\mathfrak{g}}$). Suppose not. This means that K is a number field such that $\mathcal{O}_K^{\times,+}$ lies in a proper subfield $L \subsetneq K$, so even $\mathcal{O}_K^{\times,+} \subseteq \mathcal{O}_L^{\times}$, and we get

$$(5.3) \quad s' + 2t' < s + 2 \quad s \leq s' + t' - 1$$

for the real and complex places s', t' of L by comparing ranks of rings of integers and units. This forces $t' = 0$ and $s' = s + 1$, so

$$s + 2 = [K : \mathbf{Q}] = [K : L] \cdot [L : \mathbf{Q}] = [K : L](s + 1),$$

implying $s = 0$, which is impossible since our Oeljeklaus-Toma manifolds always come from number fields with at least one real place. \square

Remark 4. Instead of identifying \mathcal{O}_K inside the fundamental group via the torsion-free quotient of the abelianization, it can also be characterized as “the largest” subgroup isomorphic to \mathbf{Z}^n for some n . See Proposition 7 for this alternative perspective.

There is also the following harmless generalization:

Proposition 8. *Let X be known to be of the shape*

$$X(K; U) := (\mathbf{H}^s \times \mathbf{C}) / (\mathcal{O}_K \rtimes U)$$

with U a finite-index subgroup of $\mathcal{O}_K^{\times,+}$ for some number field K which has $s \geq 1$ real places and precisely one complex place.

- (1) *Then $X(K; U)$ is an LCK manifold.*
- (2) *Just knowing its fundamental group π , $K^{\mathfrak{g}}$ can be reconstructed from π by the same recipe as in Prop. 1.*

³we had restricted our attention to this case in the entire text right from the beginning

(3) We have

$$\begin{aligned} & \sum_{\sigma \in \text{Gal}(K^{\mathfrak{g}}/\mathbf{Q})} \sigma U \\ &= \left\{ \alpha \in \mathcal{O}_{K^{\mathfrak{g}}}^{\times} \mid \begin{array}{l} \text{there exists } x \in \pi_{\text{ab,fr}} \text{ such that } \rho(x) \in \text{GL}_n(\mathbf{Z}) \text{ has} \\ \text{the same minimal polynomial as } \alpha. \end{array} \right\}. \end{aligned}$$

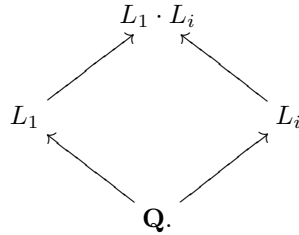
The sum on the left-hand side is the smallest subgroup of $\mathcal{O}_{K^{\mathfrak{g}}}^{\times}$ containing U and being closed under the Galois action of $K^{\mathfrak{g}}/\mathbf{Q}$.

Proof. (1) Once $\mathcal{O}_K^{\times,+}$ is admissible, any finite-index subgroup like U is as well, so $X(K;U)$ is just an instance of the construction in [OT05]. (2) The proof of Prop. 1 applies word for word, just replace each $\mathcal{O}_K^{\times,+}$ by U and whenever Dirichlet's Unit Theorem is applied, use that a finite-index subgroup of \mathbf{Z}^n must itself be isomorphic to \mathbf{Z}^n . (3) For any $\alpha \in \pi_{\text{ab,fr}} \cong U$ the minimal polynomial of $\rho(\alpha)$ matches the minimal polynomial of α as an algebraic integer. \square

Example 2. In this example we will show that for $t > 1$ complex places, the construction $X(K;U)$ of [OT05] can also produce (non-LCK) complex manifolds with isomorphic fundamental groups so that the Galois closures of their underlying number fields differ. Thus, the scope of Proposition 1 does not extend to general $X(K;U)$. To this end, consider the number fields

$$L_1 := \mathbf{Q}[S]/(S^3 + S + 1) \quad L_2 := \mathbf{Q}[T]/(T^3 - T + 2) \quad L_3 := \mathbf{Q}[T]/(T^3 - T + 1).$$

All of these fields satisfy $s = t = 1$. Henceforth, we shall also write S and T to denote the image of these elements in these fields. The element S lies in $\mathcal{O}_{L_1}^{\times}$ since its minimal polynomial $S^3 + S + 1$ has constant coefficient 1 and therefore must be a unit. We compute its single real embedding to be $-0.6823\dots$, so $S^2 \in \mathcal{O}_{L_1}^{\times,+}$ generates a subgroup isomorphic to \mathbf{Z} . One can show that $\mathcal{O}_{L_1}^{\times,+} = \mathbf{Z}\langle -S \rangle$, but we will not need to know this. Now fix $i = 2, 3$ and consider the compositum



We find that $L_1 \cdot L_i$ is a degree 9 number field with $s = 1$ real places and $t = 4$ complex places. Since T is integral, the submodule

$$(5.4) \quad J_i := \mathbf{Z}\langle 1, S, S^2, T, T^2, ST, S^2T, ST^2, S^2T^2 \rangle \subset L_1 \cdot L_i$$

defines a subring of the ring of integers and one checks that we actually have equality. As far as I can tell confirming this is the only difficult part of this computation. See Code 1 on page 31 for a verification by computer. Since our number field has precisely one real place, the admissible subgroups of $\mathcal{O}_{L_1 \cdot L_i}^{\times,+}$ (in the sense of [OT05]) have rank one and we will simply take $S^2 \in \mathcal{O}_{L_1}^{\times,+}$ as the generator. The action on the basis elements of Equation

5.4 is easy to compute, we obtain

$$(5.5) \quad \begin{array}{lll} S \cdot 1 = S & S \cdot T = ST & S \cdot S^2T = (-S-1)T \\ S \cdot S = S^2 & S \cdot T^2 = ST^2 & S \cdot ST^2 = S^2T^2 \\ S \cdot S^2 = -S-1 & S \cdot ST = S^2T & S \cdot S^2T^2 = (-S-1)T^2. \end{array}$$

The action of S^2 follows immediately. The main point here is that this table does not depend on whether $i = 2$ or 3 . For the complex manifolds $X_i := (\mathbf{H} \times \mathbf{C}^4)/(\mathcal{O}_{L_1 \cdot L_i} \rtimes \langle S^2 \rangle)$ of [OT05, §1] this table entirely determines the group structure of the semi-direct product

$$\pi_1(X_i, *) = J_i \rtimes \langle S^2 \rangle = \mathbf{Z}^9 \rtimes \mathbf{Z}.$$

Hence, X_2 and X_3 have isomorphic fundamental groups. Since they are actually classifying spaces, it follows that they even have the same homotopy type. It follows from a result of Oeljeklaus-Toma [OT05, Prop. 2.9] that the manifolds X_i *do not* admit LCK metrics. They are also concrete examples which are not ‘of simple type’ (in the sense of [OT05, Definition 1.5 and Remark 1.8]). Their underlying number fields have Galois closure $(L_1 \cdot L_i)^{\mathbf{g}} = L_1^{\mathbf{g}} \cdot L_i^{\mathbf{g}}$. It is easy to compute $L_i^{\mathbf{g}}$ for $i = 1, 2, 3$ and we find that each of them has degree 6 over \mathbf{Q} . We also find that $L_1^{\mathbf{g}} \cdot L_2^{\mathbf{g}} \cdot L_3^{\mathbf{g}}$ has degree $6^3 = 216$ over \mathbf{Q} . This implies that the degree 36 fields $L_1^{\mathbf{g}} \cdot L_2^{\mathbf{g}}$ and $L_1^{\mathbf{g}} \cdot L_3^{\mathbf{g}}$ must be different. See Code 1 on page 31 for an automated verification of this example by a computer algebra system.

Going far beyond the case of just Oeljeklaus-Toma manifolds, it is a classical theorem due to Mostow that any two compact solvmanifolds with isomorphic fundamental groups must be diffeomorphic [Mos54, Theorem A].

6. THE INVARIANT VOLUME FORM

Oeljeklaus and Toma found a very nice canonical LCK metric on $X(K; \mathcal{O}_K^{\times, +})$. See [DO98] or [Vai76] for an introduction to LCK metrics. This Hermitian metric is induced from a global Kähler potential on the universal covering space $\mathbf{H}^s \times \mathbf{C} = \{(z_1, \dots, z_{s+1}) \mid z_1, \dots, z_s \in \mathbf{H}, z_{s+1} \in \mathbf{C}\}$. The relevant Kähler potential is

$$(6.1) \quad \phi := \phi_1 + |z_{s+1}|^2$$

with

$$\phi_1 := \prod_{j=1}^s \frac{i}{(z_j - \bar{z}_j)} = \frac{1}{2^s} \prod_{j=1}^s \frac{1}{y_j},$$

where $z_j = x_j + iy_j$ for $x_j, y_j \in \mathbf{R}$, cf. [OT05], modulo the typo corrected in [PV12]. Then the underlying Riemannian metric and the associated $(1, 1)$ -form are given by

$$(6.2) \quad g = \frac{1}{2} \sum_{k,l=1}^{s+1} g_{kl} (dz_k \otimes d\bar{z}_l + d\bar{z}_l \otimes dz_k) \quad \omega = \sum_{k,l=1}^{s+1} \frac{i}{2} g_{kl} dz_k \wedge d\bar{z}_l$$

with $g_{kl} := (\partial_{z_k} \partial_{\bar{z}_l} \phi)$. One may follow the efficient explicit computation in [PV12, proof of Thm. 5.1], leading us to

$$(g_{kl}) = \begin{pmatrix} \frac{\phi_1}{2y_1^2} & \frac{\phi_1}{4y_1y_2} & \frac{\phi_1}{4y_1y_3} & \cdots & 0 \\ \frac{\phi_1}{4y_1y_2} & \frac{\phi_1}{2y_2^2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ \frac{\phi_1}{4y_1y_s} & & & \frac{\phi_1}{2y_s^2} & 0 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}.$$

In particular, the determinant of (g_{kl}) is twice the determinant of the top left $(s \times s)$ -minor. For the latter, the Leibniz formula yields

$$\begin{aligned} \det(\text{top left } (s \times s)\text{-minor}) &= \sum_{\sigma \in \Sigma_s} \text{sgn}(\sigma) 2^{\#\{j|j=\sigma(j)\}} \frac{\phi_1}{4y_1 y_{\sigma(1)}} \cdots \frac{\phi_1}{4y_s y_{\sigma(s)}} \\ &= \frac{\phi_1^s}{4^s} \left(\sum_{\sigma \in \Sigma_s} \text{sgn}(\sigma) 2^{\#\{j|j=\sigma(j)\}} \right) \frac{1}{y_1^2 \cdots y_s^2}. \end{aligned}$$

The inner bracket is easily seen to be $s+1$. One way to evaluate this is as follows: We readily see that the bracket agrees with

$$\det \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & & \vdots \\ \vdots & 1 & 2 & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix} = (-1)^s \det \left(-\text{id} - \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & 1 & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \right),$$

so the determinant is nothing but $(-1)^s p_A(-1)$ with $p_A(t) := \det(t - A)$ the characteristic polynomial of the matrix A whose entries are all 1. The latter has the single non-zero eigenvector $(1, 1, \dots, 1)^t$ with eigenvalue s . Therefore, the characteristic polynomial is $(t - s)t^{s-1}$. Hence, $(-1)^s p_A(-1) = (-1 - s)(-1)^{2s-1} = (s+1)$, proving the claim. Returning to our main computation,

$$(6.3) \quad \det(g_{kl}) = 2 \cdot \frac{(s+1)}{4^s} \frac{1}{y_1^2 \cdots y_s^2} \phi_1^s = \frac{(s+1)}{2^{2s+s^2-1}} \frac{1}{y_1^{s+2} \cdots y_s^{s+2}}.$$

The Kähler potential of Equation 6.1 defines a genuine Kähler form on $\mathbf{H}^s \times \mathbf{C}$. One easily checks that translations from \mathcal{O}_K leave it invariant, while the multiplication with elements $\alpha \in \mathcal{O}_K^{\times,+}$ changes it by a homothety. More precisely, from Equation 6.2 we get

$$\omega = \frac{i}{2^{s+3} y_1 \cdots y_s} \left(\sum_{k,l=1}^s \frac{2^{\delta_{k=l}}}{y_k y_l} dz_k \wedge d\bar{z}_l \right) + i dz_{s+1} \wedge d\bar{z}_{s+1}$$

and therefore

$$\begin{aligned} \alpha^* \omega &= \frac{i}{2^{s+3} \sigma_1(\alpha) \cdots \sigma_s(\alpha) y_1 \cdots y_s} \left(\sum_{k,l=1}^s \frac{2^{\delta_{k=l}}}{\sigma_k(\alpha) \sigma_l(\alpha) y_k y_l} \sigma_k(\alpha) \overline{\sigma_l(\alpha)} dz_k \wedge d\bar{z}_l \right) \\ &\quad + i |\sigma_{s+1}(\alpha)|^2 dz_{s+1} \wedge d\bar{z}_{s+1}. \end{aligned}$$

Here α^* denotes the pullback along the action of $\alpha \in \mathcal{O}_K^{\times,+}$. Note that $\prod_{j=1}^{s+2} \sigma_j(\alpha) = N(\alpha) = +1$ (usually ± 1 since it is a unit, but $+1$ since all real embeddings $\sigma_j(\alpha) > 0$ are positive by assumption and the remaining factor is $|\sigma_{s+1}(\alpha)|^2 = |\sigma_{s+1}(\alpha) \sigma_{s+2}(\alpha)|$ of the last pair of complex embeddings; this is also positive). Hence, $\frac{1}{\sigma_1(\alpha) \cdots \sigma_s(\alpha)} = |\sigma_{s+1}(\alpha)|^2$, thus $\alpha^* \omega = |\sigma_{s+1}(\alpha)|^2 \omega$. The group $\mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}$ acts by homotheties on the honest Kähler form ω , therefore it does not descend to the quotient and will not equip it with a Kähler metric itself, but it means that the quotient is at least locally conformally Kähler (LCK). The relevant invariant form is

$$\tilde{\omega} := y_1 \cdots y_s \omega \quad \text{i.e.} \quad \tilde{\omega} := e^f \omega \text{ with } f := \log(y_1 \cdots y_s),$$

i.e. this form is invariant under the group action, descends to the Oeljeklaus-Toma manifold, but is not Kähler anymore. We compute $\alpha^*\tilde{\omega} = \tilde{\omega}$ and $d\tilde{\omega} = df \wedge \tilde{\omega}$. In particular, the Lee form [DO98, Ch. 1] of an Oeljeklaus-Toma manifold is

$$\theta := df = d\log(y_1 \cdots y_s) = \sum_{j=1}^s d\log(y_j) = \sum_{j=1}^s \frac{dy_j}{y_j}.$$

Remark 5 (Inoue surface of type S^0). In the case $s = 1$ this simplifies to

$$\tilde{\omega} = \frac{i}{8} \frac{dz_1 \wedge d\bar{z}_1}{y_1^2} + iy_1 dz_2 \wedge d\bar{z}_2,$$

which is essentially the $(1,1)$ -form associated to the Tricerri metric.

The Oeljeklaus-Toma manifold comes with a canonical volume form. Just like $\mathbf{H}^s \times \mathbf{C}$ carries the Kähler form ω and an invariant form $\tilde{\omega}$ associated to the LCK metric, $\mathbf{H}^s \times \mathbf{C}$ has a canonical volume form vol from ω , and an invariant counterpart \widetilde{vol} belonging to $\tilde{\omega}$. The volume form can be computed for example by

$$vol = \frac{\omega^{s+1}}{(s+1)!} = \left(\frac{i}{2}\right)^{s+1} \det(g_{kl}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{s+1} \wedge d\bar{z}_{s+1},$$

which we can unravel further thanks to Equation 6.3. We may now either switch to $\tilde{\omega}$, or we can equivalently work with the scaled metric $(y_1 \cdots y_s g_{kl})$ instead of (g_{kl}) . Then the determinant scales to $\det(y_1 \cdots y_s g_{kl}) = (y_1 \cdots y_s)^{s+1} \det(g_{kl})$ since (g_{kl}) is a $(s+1) \times (s+1)$ -matrix. Hence, we obtain

$$\widetilde{vol} := \left(\frac{i}{2}\right)^{s+1} \frac{(s+1)}{2^{2s+s^2-1}} \frac{1}{y_1 \cdots y_s} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{s+1} \wedge d\bar{z}_{s+1},$$

which we may rewrite as

$$(6.4) \quad \widetilde{vol} = \frac{(s+1)}{2^{2s+s^2-1}} \frac{1}{y_1 \cdots y_s} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{s+1} \wedge dy_{s+1}.$$

Note that by writing $\frac{dy}{y} = d\log y$, this looks just like the Euclidean volume form in suitable coordinates, say $y := e^r$ or e^{2r} . This is the key reason why Prop. 3 will turn out to be true.

$$= \frac{(s+1)}{2^{2s+s^2-1}} \cdot \bigwedge_{j=1}^s (dx_j \wedge d\log(y_j)) \wedge dx_{s+1} \wedge dy_{s+1}.$$

In the next section we shall tailor a fundamental domain suitable to this volume form.

7. A FUNDAMENTAL DOMAIN

Fix a number field K with $s \geq 1$ real places and precisely one complex place.

In this section we shall determine a fundamental domain for the action of $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ on $\mathbf{H}^s \times \mathbf{C}$. By Lemma 1 this action is precisely the same as the action of the (almost) “standard Borel”

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subset \mathrm{SL}_2(\mathcal{O}_K) \quad (\text{with } a \in \mathcal{O}_K^{\times,+}, b \in \mathcal{O}_K).$$

Here $\mathrm{SL}_2(\mathcal{O}_K)$ acts under the diagonal embeddings of Equation 3.3. On the individual factors \mathbf{H} this is precisely the Möbius action. We will therefore work with the coordinates

coming from the Iwasawa decomposition of $\mathrm{SL}_2(\mathbf{R})$, see Equation 2.1. It parametrizes \mathbf{H} in exactly this shape, namely $\mathbf{H} \simeq A \cdot N$. We use the following explicit coordinates:

$$(7.1) \quad \begin{pmatrix} e^r & b \\ & e^{-r} \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ & \frac{1}{\sqrt{y}} \end{pmatrix}$$

with $r, b \in \mathbf{R}$, giving a semi-direct product presentation

$$(7.2) \quad \begin{aligned} 0 &\longrightarrow \mathbf{R} \longrightarrow A \cdot N \longrightarrow \mathbf{R} \longrightarrow 0 \\ b &\mapsto \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} e^r & b \\ & e^{-r} \end{pmatrix} \mapsto r. \end{aligned}$$

Replicating the decomposition stemming from the coordinates of Equation 7.2 for each real place, we get

$$(7.3) \quad \begin{array}{ccccccc} \mathbf{R}^s \times \mathbf{C} & \longrightarrow & \mathbf{R} & \times \cdots \times & \mathbf{R} & \times & \mathbf{C} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{H}^s \times \mathbf{C} & \longrightarrow & AN & \times \cdots \times & AN & \times & \mathbf{C} \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{R}^s & \longrightarrow & \mathbf{R} & \times \cdots \times & \mathbf{R} & & \end{array}$$

This picture is also related to the solvmanifold viewpoint proposed by Kasuya [Kas13]. Under the diagonal embedding of Equation 3.3, we have $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2 \subset \prod_{j=1}^s AN \rtimes \mathrm{Aut} \mathbf{C}$, and on the factors AN this group action is just matrix multiplication, see Lemma 1. Moreover, the arrow

$$(7.4) \quad \begin{array}{ccc} \mathbf{H}^s \times \mathbf{C} & \begin{pmatrix} e^{r_i} & b_i \\ & e^{-r_i} \end{pmatrix}_{i=1,\dots,s} \times (b_{s+1}) \\ \downarrow & \downarrow \\ \mathbf{R}^s & (r_1, \dots, r_s) \end{array}$$

is $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ -equivariant, where the action on the bottom row unravels to factor through

$$\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2 \twoheadrightarrow (\mathcal{O}_K^{\times,+})^2$$

and $\alpha \in (\mathcal{O}_K^{\times,+})^2$ is easily seen to act as translations

$$\alpha \cdot (r_1, \dots, r_s) = (\log |\sigma_1 \alpha| + r_1, \dots, \log |\sigma_s \alpha| + r_s).$$

By Dirichlet's Unit Theorem we can pick free generators $(\mathcal{O}_K^{\times,+})^2 = \mathbf{Z} \langle \varepsilon_1, \dots, \varepsilon_s \rangle \simeq \mathbf{Z}^s$, i.e. a multiplicatively independent system of units in this group. Then

$$\begin{aligned} \Lambda &:= \left\{ \sum_{i=1}^s \beta_i B_i \mid 0 \leq \beta_i < 1 \right\} \\ B_i &:= (\log |\sigma_1(\varepsilon_i)|, \dots, \log |\sigma_s(\varepsilon_i)|)^t \end{aligned}$$

is a fundamental domain for the action of $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ on the base row of Diagram 7.3. Next, suppose we are given an element in the middle row, say $(b_1, \dots, b_s, b_{s+1}, r_1, \dots, r_s)$ with $b_1, \dots, b_s \in \mathbf{R}$, $b_{s+1} \in \mathbf{C}$, $r_1, \dots, r_s \in \mathbf{R}$. Then for *fixed* r_1, \dots, r_s an element $\alpha \in \mathcal{O}_K \subset \mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ is easily checked to act as

$$\begin{pmatrix} 1 & \sigma_i \alpha \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{r_i} & b_i \\ & e^{-r_i} \end{pmatrix} = \begin{pmatrix} e^{r_i} & b_i + e^{-r_i} \sigma_i(\alpha) \\ & e^{-r_i} \end{pmatrix}$$

in the i -th coordinate. In particular, the orbit under \mathcal{O}_K stays in the same fiber over r_1, \dots, r_s . Fixing the fiber, we see that \mathcal{O}_K acts solely on the coordinates $b_1, \dots, b_s \in \mathbf{R}$

and $b_{s+1} \in \mathbf{C}$ by translation. Moreover, if we pick generators $\mathcal{O}_K = \mathbf{Z} \langle a_1, \dots, a_{s+2} \rangle \simeq \mathbf{Z}^{s+2}$ a fundamental domain for the action of \mathcal{O}_K in the fiber over r_1, \dots, r_s is given by

$$(7.5) \quad \Phi(r_1, \dots, r_s) := \left\{ \sum_{i=1}^{s+2} \alpha_i \tilde{A}_i \mid 0 \leq \alpha_i < 1 \right\}$$

with $\tilde{A}_i := (e^{-r_1} \sigma_1(a_i), \dots, e^{-r_s} \sigma_s(a_i), \sigma_{s+1}(a_i))^t$.

Proposition 9. *The set*

$$\begin{aligned} \mathcal{Fund} &:= \left\{ \coprod_{(r_1, \dots, r_s) \in \Lambda} \Phi(r_1, \dots, r_s) \right\} \\ &= \left\{ (r_1, \dots, r_s, b_1, \dots, b_{s+1}) \mid \begin{array}{l} r_1, \dots, r_s \in \Lambda \\ b_1, \dots, b_{s+1} \in \Phi(r_1, \dots, r_s) \end{array} \right\} \end{aligned}$$

is a fundamental domain for the action of $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ on $\mathbf{H}^s \times \mathbf{C}$ in (r_i, b_i) -coordinates.

Proof of Prop. 9. We prove that the inclusion

$$\mathcal{Fund} \hookrightarrow (\mathbf{R}^s \times \mathbf{C}) \times \mathbf{R}^s$$

induces a bijection onto the quotient by $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$.

(*Surjectivity*) We have already observed that the downward arrow in Diagram 7.4 is equivariant. By Dirichlet's Unit Theorem the s different vectors

$$B'_i := (\log |\sigma_1(\varepsilon_i)|, \dots, \log |\sigma_s(\varepsilon_i)|, \log |\sigma_{s+1}(\varepsilon_i)|)^t$$

for $i = 1, \dots, s$ give a full rank lattice in the “log-norm” hyperplane

$$H := \{(v_1, \dots, v_{s+1}) \mid v_1 + \dots + v_s + 2v_{s+1} = 0\} \subseteq \mathbf{R}^{s+1}.$$

Thus, there is a linear isomorphism

$$\mathbf{R}^s \rightarrow H$$

$$\begin{aligned} (v_1, \dots, v_s) &\mapsto (v_1, \dots, v_s, -\frac{1}{2}(v_1 + \dots + v_s)) \\ (v_1, \dots, v_s) &\leftarrow (v_1, \dots, v_s, v_{s+1}), \end{aligned}$$

where \mathbf{R}^s is understood to refer to the base in Diagram 7.4. It follows that the s vectors $B_i := (\log |\sigma_1(\varepsilon_i)|, \dots, \log |\sigma_s(\varepsilon_i)|)^t$, i.e. just the image of the B'_i under this isomorphism, span a full rank lattice in \mathbf{R}^s . Hence, since the B_i are thus an \mathbf{R} -vector space basis, each element in \mathbf{R}^s has a unique presentation as

$$\sum_{i=1}^s (n_i + \beta_i) B_i \quad \text{with} \quad n_i \in \mathbf{Z}, 0 \leq \beta_i < 1.$$

Thus, letting $\alpha := \varepsilon_1^{-n_1} \dots \varepsilon_s^{-n_s} \in (\mathcal{O}_K^{\times,+})^2 \subset \mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ act, we obtain an element of the orbit which lies in our fundamental domain Λ for the base. Obviously, since our map is equivariant, we can let the same uniquely determined element act on the entire space. Thus, we have found a representative of our element in $(\mathbf{R}^s \times \mathbf{C}) \times \Lambda$. Next, the translation action of \mathcal{O}_K leaves the base invariant and just acts in the fibers of Equation 7.3. We get a unique presentation

$$\sum_{i=1}^{s+2} (m_i + \alpha_i) \tilde{A}_i \quad \text{with} \quad m_i \in \mathbf{Z}, 0 \leq \alpha_i < 1,$$

thus, letting $\beta := \sum m_i b_i$ act for $\mathcal{O}_K = \mathbf{Z}\langle a_1, \dots, a_{s+2} \rangle$, we get a unique representative in $\Phi(r_1, \dots, r_s) \times \{(r_1, \dots, r_s)\} \in \mathcal{F}und$, as desired. Note that the group elements we acted by were canonically determined, so we actually get a well-defined map

$$(\mathbf{R}^s \times \mathbf{C}) \times \mathbf{R}^s \longrightarrow \mathcal{F}und.$$

(*Injectivity*) Suppose $x, y \in \mathcal{F}und$ lie in the same orbit of the action of $\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$. By the equivariance of the morphism in Diagram 7.4 it follows that their images in \mathbf{R}^s lie in the same orbit of the action of $(\mathcal{O}_K^{\times,+})^2$ on \mathbf{R}^s . But since $x, y \in \mathcal{F}und$, their images $\bar{x}, \bar{y} \in \mathbf{R}^s$ lie in Λ , and since this was a fundamental domain for $(\mathcal{O}_K^{\times,+})^2$ we must have $\bar{x} = \bar{y}$. But then x, y lie in the same fiber $\Phi(r_1, \dots, r_s)$. We check that $\mathcal{O}_K \subseteq \mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2$ is the largest subgroup stabilizing a fiber, which implies that x, y only differ by the translation action of \mathcal{O}_K inside the fiber. But $\Phi(r_1, \dots, r_s)$ was constructed as a fundamental domain for this action, so we deduce $x = y$. \square

We may restate this in more conventional coordinates. Define (or recall) the standard Minkowski fundamental domain

$$\Phi_{\text{Mink}} := \left\{ \sum_{i=1}^{s+2} \alpha_i \tilde{A}_i \mid 0 \leq \alpha_i < 1 \right\} \subseteq \mathbf{R}^s \times \mathbf{C}$$

with $\tilde{A}_i^* := (\sigma_1(a_i), \dots, \sigma_s(a_i), \sigma_{s+1}(a_i))^t$. Just from a change of coordinates Prop. 9 can equivalently be reformulated as follows:

Corollary 1. *The set $\Lambda \times \Phi_{\text{Mink}} =$*

$$\left\{ (x_1, y_1, \dots, x_s, y_s, x_{s+1} + iy_{s+1}) \mid \begin{array}{l} \frac{1}{2} \log y_1, \dots, \frac{1}{2} \log y_s \in \Lambda \\ x_1, \dots, x_s, x_{s+1} + iy_{s+1} \in \Phi_{\text{Mink}} \end{array} \right\}$$

is the same fundamental domain, but in (x_i, y_i) -coordinates.

8. THE VOLUME COMPUTATION

Proof of Prop. 3. Let us compute the volume of $X := X(K; \mathcal{O}_K^{\times,+})$. For this we will integrate its canonical volume form \widetilde{vol} on X , which is best done by integrating it over our fundamental domain of Cor. 1. We compute

$$\begin{aligned} \int_X \widetilde{vol} &= \frac{1}{2^s} \int_{(\mathbf{H}^s \times \mathbf{C}) / (\mathcal{O}_K \rtimes (\mathcal{O}_K^{\times,+})^2)} \widetilde{vol} = \frac{1}{2^s} \int_{\Lambda \times \Phi_{\text{Mink}}} \widetilde{vol} \\ &= \frac{1}{2^s} \int_{\Lambda \times \Phi_{\text{Mink}}} \frac{(s+1)}{2^{2s+s^2-1}} \frac{1}{y_1 \dots y_s} dx_1 \wedge dy_1 \wedge \dots \wedge dx_{s+1} \wedge dy_{s+1}. \end{aligned}$$

Switching to r -coordinates, i.e. substituting $y_i = e^{2r_i}$ for $i = 1, \dots, s$, this effectively reduces to computing an Euclidean volume, namely

$$\begin{aligned} &= \frac{1}{2^s} \frac{(s+1)}{2^{2s+s^2-1}} \int_{\Lambda \times \Phi_{\text{Mink}}} dx_1 \wedge dr_1 \wedge \dots \wedge dx_s \wedge dr_s \wedge dx_{s+1} \wedge dy_{s+1} \\ &= \frac{1}{2^s} \frac{(s+1)}{2^{2s+s^2-1}} \left(\int_{\Lambda} dx_1 \wedge \dots \wedge dx_s \wedge dx_{s+1} \wedge dy_{s+1} \right) \left(\int_{\Phi_{\text{Mink}}} dr_1 \wedge \dots \wedge dr_s \right) \\ &= \frac{1}{2^s} \frac{(s+1)}{2^{2s+s^2-1}} \cdot \det(\tilde{A}_1^*, \dots, \tilde{A}_{s+2}^*) \cdot \det(B_1, \dots, B_s). \end{aligned}$$

Now we can use the classical fact that the vectors $\tilde{A}_1^*, \dots, \tilde{A}_{s+2}^*$, which are generating the Minkowski fundamental domain, span a parallelepiped of Euclidean volume $2^{-t} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|}$ with t the number of complex embeddings. Moreover, the determinant of the vectors B_1, \dots, B_s is almost literally the definition of the Dirichlet regulator:

$$\begin{aligned} &= \frac{1}{2^s} \frac{(s+1)}{2^{2s+s^2-1}} \cdot \left(\frac{1}{2} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \right) \cdot (2^s \cdot R_K) \\ &= \frac{(s+1)}{2^{2s+s^2}} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \cdot R_K. \end{aligned}$$

The factor 2^s in front of the regulator R_K occurs as follows: We would get precisely the regulator here if $\varepsilon_1, \dots, \varepsilon_s$ was a basis for \mathcal{O}_K^\times . However, our $\varepsilon_1, \dots, \varepsilon_s$ are a basis for $(\mathcal{O}_K^{\times,+})^2$. We recall the analytic class number formula, stating that (in the case we consider)

$$\text{res}_{s=1} \zeta_K(s) = \frac{2^s \pi h_K R_K}{\sqrt{|\Delta_{K/\mathbf{Q}}|}}.$$

Solving for R_K yields the claim by plugging it into our previous formula for the volume. We get Equation 0.3 as desired. \square

One can actually ‘speed up’ this computation slightly by working directly with a fundamental domain under the action of the full group $\mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}$, leading the two mutually cancelling factors $\frac{1}{2^s}$ and 2^s to disappear altogether.

Example 3. Consider the cubic field

$$(8.1) \quad K := \mathbf{Q}[T]/(T^3 + T^2 - 1).$$

The image $\overline{T} \in K$ is actually a generator of $\mathcal{O}_K^{\times,+}$ because its norm is one and its single real embedding has value $0.7548\dots > 0$. The number field K has discriminant $\Delta_{K/\mathbf{Q}} = -23$, class number $h_K = 1$ and regulator

$$R_K = |\log 0.754877\dots| = 0.28119957432\dots$$

It has one real and one complex place. We may form its Oeljeklaus-Toma manifold

$$X := X(K; \mathcal{O}_K^{\times,+}),$$

giving a (non-Kähler) Inoue surface of type S^0 with Tricerri’s metric. According to Prop. 3 its volume is

$$\text{Vol}(X) = \frac{1}{4} \cdot \sqrt{23} \cdot 0.28119957432\dots \approx 0.3371\dots$$

We will show in Prop. 12 that no smaller volume is possible among cubic fields. The commutator subgroup of its fundamental group is (by Prop. 5)

$$[\pi, \pi] = J(\mathcal{O}_K^{\times,+}) = (1 - \overline{T}),$$

the ideal in \mathcal{O}_K generated by $1 - \overline{T}$. From the minimal polynomial, Equation 8.1, we see that $\overline{T}^3 + \overline{T}^2 = 1$ and a simple polynomial division reveals that $(1 - \overline{T}) \cdot (\overline{T}^2 + 2\overline{T} + 2) = 1$, showing that $J(\mathcal{O}_K^{\times,+}) = (1)$ is actually the entire ring of integers. So the maximal abelian quotient $\pi_{\text{ab}} \simeq \mathbf{Z}$ is already torsion-free itself. By Prop. 2 we therefore have

$$H_1(X, \mathbf{Z}) = \mathbf{Z}.$$

Following the recipe of Prop. 1 we let a generator act on \mathbf{Z}^3 and this will be \overline{T} or \overline{T}^{-1} . We have no way of distinguishing them if we are just given \mathbf{Z} abstractly. Say it was \overline{T} , and we

get precisely that its action on \mathbf{Z}^3 has minimal polynomial $x^3 + x^2 - 1$ in $\mathbf{Z}[x]$, generating K over \mathbf{Q} . Adjoining all three complex roots yields $K^{\mathfrak{g}}/\mathbf{Q}$, a field of degree 6.

9. PRESCRIBED TORSION

We want to exhibit a particularly nice family of Inoue surfaces for which we can freely prescribe the order of the torsion in $H_1(X, \mathbf{Z})$. As will be clear from the proof, this construction largely rests on ideas of Ishida, porting from number theory to geometry.

Proposition 10. *For any given $m \geq 1$ there exists an Inoue surface X of type S^0 with*

$$H_1(X, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/m$$

and equipped with the Oeljeklaus-Toma metric, it has volume

$$\text{Vol}(X) = \frac{1}{4} \cdot \sqrt{4m^3 + 27} \cdot \log \left| z - \frac{m}{3z} \right|$$

for the real number

$$z := \sqrt[3]{\frac{1}{2} + \frac{\sqrt{3}}{18} \sqrt{4m^3 + 27}}.$$

In fact, X can be constructed as a finite unramified covering

$$(9.1) \quad \begin{array}{c} X \\ \downarrow \\ X(K; \mathcal{O}_K^{\times,+}) \end{array}$$

of the Oeljeklaus-Toma manifold

$$(9.2) \quad X := X(K; \mathcal{O}_K^{\times,+}) \quad \text{for} \quad K := \mathbf{Q}[T]/(T^3 + mT - 1).$$

If $4m^3 + 27$ is square-free, this covering is trivial. Alternatively, suppose $m = 3k$ and $4k^3 + 1$ is square-free: Then if $3 \nmid k$, the covering is also trivial. If $3 \mid k$, it is a covering of degree 3.

I suspect that all $H_1(X, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/m$ can be realized by genuine Oeljeklaus-Toma manifolds without the need to allow finite coverings.

Proof. Let $m \geq 1$ be given. The polynomial $T^3 + mT - 1$ has one sign change in its coefficients, so by Descartes Sign Rule it has a single positive real root and no negative real roots. Moreover, it is irreducible over \mathbf{Q} (*Proof:* Otherwise it has a rational root α_1 . Hence, over the algebraic closure it factors as $(T - \alpha_1)(T - \alpha_2)(T - \alpha_3)$ with $\alpha_1 \in \mathbf{Q} \cap \overline{\mathbf{Z}} = \mathbf{Z}$ and $\alpha_2, \alpha_3 \in \overline{\mathbf{Z}}$. Since $\alpha_1 \alpha_2 \alpha_3 = 1$ it follows that α_1 is also a unit, so $\alpha_1 = 1$ since we already know that there is no negative real root. But by plugging in we see that this is certainly not a root). It follows that

$$K := \mathbf{Q}[T]/(T^3 + mT - 1)$$

is a cubic number field with $s = t = 1$. We write \overline{T} to denote the image of T in K . Since the constant coefficient in the minimal polynomial of \overline{T} is -1 , it is a unit in \mathcal{O}_K^{\times} and we had already seen that its single real embedding is necessarily positive. Thus, $\overline{T} \in \mathcal{O}_K^{\times,+}$ and it generates a subgroup $U := \mathbf{Z} \langle \overline{T} \rangle$ of finite index. Similarly, instead of the full ring of integers we so far just understand $\mathbf{Z}[\overline{T}] \subseteq \mathcal{O}_K$, which might be of some finite index, too.

This elementary construction already allows us to construct X : We consider the complex manifold X , defined by

$$(9.3) \quad \begin{array}{c} \frac{\mathbf{H} \times \mathbf{C}}{\mathbf{Z}[\overline{T}] \rtimes U} = X \\ \downarrow \\ \frac{\mathbf{H} \times \mathbf{C}}{\mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}} = X(K, \mathcal{O}_K^{\times,+}) \end{array}$$

and equip it with the Oeljeklaus-Toma metric, which is of course also invariant under the action since $\mathbf{Z}[\overline{T}] \rtimes U$ forms some finite index subgroup of $\mathcal{O}_K \rtimes U$. In particular, X is compact as well. It clearly is an Inoue surface. The definition of the ideal J (Definition 1) also makes sense in the subring $\mathbf{Z}[\overline{T}] \subset \mathcal{O}_K$ and we compute

$$(9.4) \quad \begin{aligned} J(U) &= \mathbf{Z}[\overline{T}] / (\overline{T} - 1) \\ &= \mathbf{Z}[T] / (T - 1, T^3 + mT - 1) = \mathbf{Z}/m. \end{aligned}$$

We leave it to the reader to check that Prop. 6 can be generalized to the manifold X and gives us $H_1(X, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/m$. In fact, the proof carries over verbatim. Finally, we can compute its volume as follows: Instead of the discriminant $\Delta_{K/\mathbf{Q}}$ of the number field K , we now just get the discriminant of the order $\mathbf{Z}[\overline{T}] \subset \mathcal{O}_K$, but this makes things easier since that is just the discriminant of the generating polynomial, i.e. $-4m^3 - 27$. The regulator matrix for K is the (1×1) -matrix with the single entry $\log |\sigma_1(\overline{T})|$, where $\sigma_1(\overline{T})$ denotes the single real embedding of \overline{T} , or equivalently the single real root of $T^3 + mT - 1$. We may solve this using the classical Vieta substitution t for depressed cubics (a variant to the Cardano-Tartaglia formula): The real root is given by

$$t := z - \frac{m}{3z} \quad \text{for} \quad z := \sqrt[3]{\frac{1}{2} + \frac{\sqrt{3}}{18} \sqrt{4m^3 + 27}}.$$

This formula is ‘fairly’ simple since in the polynomial $T^3 + mT - 1$ the quadratic term is already eliminated.

The rest of the proof, and the only difficult part, exclusively concerns the question to control the index of

$$\mathbf{Z}[\overline{T}] \rtimes U \subseteq \mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}$$

in order to understand the degree of the covering. For the discriminant of the order $\mathbf{Z}[\overline{T}] \subseteq \mathcal{O}_K$ we compute $\text{disc}(T^3 + mT - 1) = -4m^3 - 27$ and therefore

$$(9.5) \quad -4m^3 - 27 = \Delta_{K/\mathbf{Q}} \cdot [\mathcal{O}_K : \mathbf{Z}[\overline{T}]]^2$$

by the discriminant-index formula. Hence, if $4m^3 + 27$ is square-free, we must have $[\mathcal{O}_K : \mathbf{Z}[\overline{T}]] = 1$ and therefore $\mathbf{Z}[\overline{T}] = \mathcal{O}_K$. Next, we use a clever theorem of Ishida telling us that this also implies that \mathcal{O}_K^{\times} is generated by \overline{T} , namely [Ish73, Theorem 1] (strictly speaking, Ishida’s theorem only applies for $m \geq 2$, so we ask the reader to deal with the single case $m = 1$ either by using a computer – or by hand. The latter can be done by checking that the norm equation $N(-) = +1$ cannot have a real solution of smaller absolute value). The fundamental unit \overline{T} must moreover be totally positive since it was chosen from a polynomial which did not have negative real roots. Thus,

$$(9.6) \quad \mathbf{Z}[\overline{T}] \rtimes U = \mathcal{O}_K \rtimes \mathcal{O}_K^{\times,+}$$

and the manifold X of Equation 9.3 becomes literally a genuine Oeljeklaus-Toma manifold. Let us deal with the remaining case: $m = 3k$ and $4k^3 + 1$ is square-free. The same theorem

of Ishida [Ish73, Theorem 1] tells us that this also suffices to have $U = \mathcal{O}_K^{\times,+}$. However, $[\mathcal{O}_K : \mathbf{Z}[\overline{T}]]$ can be larger than one. Actually, the paper of Ishida gives us also all the tools we need to deal with this problem, but Ishida does not summarize his findings in this case as a separate theorem, so let me guide you through his argument: In [Ish73, §3, all on page 248] he first deduces from the discriminant-index formula, i.e. Equation 9.5, that

$$[\mathcal{O}_K : \mathbf{Z}[\overline{T}]] = 3^d$$

for some $d \geq 0$. In the case that $3 \nmid k$, he uses that \overline{T} and $\overline{T} + 1$ clearly generate the same number field, but

$$(T+1)^3 + m(T+1) - 1 = T^3 + 3T^2 + (3k+3)T + 3k$$

is an Eisenstein polynomial at the prime $p = 3$, which implies that $3 \nmid [\mathcal{O}_K : \mathbf{Z}[\overline{T}]]$. Thus, again $\mathbf{Z}[\overline{T}] = \mathcal{O}_K$ and we are back in the situation of Equation 9.6. It remains to deal with the case $3 \mid k$, so $3^3 \mid m$. In this case Ishida exhibits the element

$$\frac{1}{3}(1 + \overline{T} + \overline{T}^2) \in \frac{1}{3}\mathbf{Z}[\overline{T}],$$

which can be checked by direct computation to be integral, i.e. it lies in \mathcal{O}_K . This forces $d \geq 1$ and using the discriminant-index formula once more, he concludes $[\mathcal{O}_K : \mathbf{Z}[\overline{T}]] = 3$. Thus, our covering is also of degree 3. \square

Example 4. With the help of the computer we can compute the index of $\mathbf{Z}[\overline{T}]$ inside \mathcal{O}_K , resp. U inside $\mathcal{O}_K^{\times,+}$. Several of the cases below are of course fully explained by the proposition above. However, not all of them, and in particular we see that the covering of Equation 9.1 can sometimes have fairly large degree:

m	$[\mathcal{O}_K : \mathbf{Z}[\overline{T}]]$	$[\mathcal{O}_K^{\times,+} : \mathbf{Z}[\overline{T}]]$	$x^2 \mid 4m^3 + 27$
8	5	2	5^2
16	1	1	
24	1	1	3^4
32	1	1	
40	1	1	
48	1	1	3^2
56	31	2	31^2
64	1	1	
72	$3 \cdot 11$	2	$3^2 \cdot 11^2$

The rightmost column lists square factors. Among the first 500 values of m we get $\mathcal{O}_K = \mathbf{Z}[\overline{T}]$ for 415 of them. The condition for $4m^3 + 27$ to be square-free gives a reasonable sufficient condition to have $[\mathcal{O}_K : \mathbf{Z}[\overline{T}]] = 1$, but is still quite remote from a precise criterion. As I have learnt from Ishida's paper [Ish73], it was shown by the famous Erdős that $4m^3 + 27$ is square-free for infinitely many m .

10. A CURIOSITY

As we had seen from §4 the structure of the ideal $J(U)$ can be quite a non-trivial matter. Even though its concrete structure seems fairly elusive from the outset, one can bound its index in terms of the units of the underlying number field. Sadly, controlling their size is similarly inaccessible. However, these two elusive bounds control each other.

I only record the following estimate as a curiosity. Since I know of no way to compute the volume of an Oeljeklaus-Toma manifold except from the arithmetic invariants of the

underlying number field, I would not know how to put the following inequality into any computational use.

Proposition 11. *Let K be a number field with $s = t = 1$. Then the torsion in the first homology of the Oeljeklaus-Toma surface $X := X(K; \mathcal{O}_K^{\times,+})$ can be bounded in terms of the volume and discriminant. Specifically,*

$$\#H_1(X, \mathbf{Z})_{\text{tor}} \leq 3(z + z^2)$$

where

$$z := \max(w, \sqrt{1/w}) \quad \text{and} \quad w := \exp \left(4 \frac{\text{Vol}(X)}{\sqrt{|\Delta_{K/\mathbf{Q}}|}} \right).$$

Proof. From Prop. 6 we have the equality

$$\#H_1(X, \mathbf{Z})_{\text{tor}} = \#(\mathcal{O}_K / J(\mathcal{O}_K^{\times,+})).$$

By Dirichlet's Unit Theorem $\mathcal{O}_K^{\times} \simeq \langle -1 \rangle \times \mathbf{Z} \langle u \rangle$ with u a fundamental unit. Without loss of generality we can assume that u is totally positive, otherwise replace u by $-u$. Then u is a generator of $\mathcal{O}_K^{\times,+}$. By Lemma 2 we therefore have

$$J(\mathcal{O}_K^{\times,+}) = (1 - u).$$

As this is a principal ideal, its ideal norm can be computed just in terms of the norm of the generating element. This means that

$$\#(\mathcal{O}_K / J(\mathcal{O}_K^{\times,+})) = |N_{K/\mathbf{Q}}(1 - u)| = \prod_{i=1}^3 \sigma_i(1 - u).$$

As usual, let σ_1 denote the single real embedding and $\sigma_2, \overline{\sigma_2} = \sigma_3$ are the complex conjugate embeddings of the single complex place. We have $\sigma_1(u) > 0$ and therefore $N_{K/\mathbf{Q}}(u) = \sigma_1(u) |\sigma_2(u)|^2 > 0$ and the norm lies in $\{\pm 1\} = \mathbf{Z}^{\times}$ since u is a unit. Hence, $N_{K/\mathbf{Q}}(u) = 1$ and we can continue the above computation with

$$\begin{aligned} &= 1 - \sum_{i=1}^3 \sigma_i(u) + \sum_{1 \leq i < j \leq 3} \sigma_i(u) \sigma_j(u) - N_{K/\mathbf{Q}}(u) \\ (10.1) \quad &= \sum_{i < j} \sigma_i(u) \sigma_j(u) - \sum_{i=1}^3 \sigma_i(u). \end{aligned}$$

By $\sigma_1(u) |\sigma_2(u)|^2 = 1$ we have $|\sigma_2(u)| = \sqrt{1/\sigma_1(u)}$. Thus, for $z := \max(\sigma_1(u), \sqrt{1/\sigma_1(u)}) > 0$ we get the estimate

$$\#(\mathcal{O}_K / J(\mathcal{O}_K^{\times,+})) \leq 3z^2 + 3z = 3(z + z^2)$$

from Equation 10.1. From Prop. 3 we know that

$$\text{Vol}(X) = \frac{1}{4} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \cdot R_K = \frac{1}{4} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \cdot \log |\sigma_1 u|,$$

since the Dirichlet regulator is just formed from a (1×1) -matrix in the present situation, and the single entry comes from the logarithmic embedding of the fundamental unit. Thus, reversing the usual logic, we can also say that

$$\sigma_1 u = \exp \left(4 \frac{\text{Vol}(X)}{\sqrt{|\Delta_{K/\mathbf{Q}}|}} \right).$$

The claim follows from connecting this with our previous upper bound. \square

Example 5. As usual in this text, let us compare this estimate to precise values: We shall study the number fields

$$K := \mathbf{Q}[T]/(T^3 + 8T - m)$$

for $m \geq 1$, whenever the given polynomial is irreducible. It is easy to see that these are number fields with $s = t = 1$. The discriminant of the order $\mathbf{Z}[\overline{T}] \subseteq \mathcal{O}_K$ is easily computed to be

$$\Delta_{\mathbf{Z}[\overline{T}]/\mathbf{Q}} = -27m^2 - 2048$$

and for most of the $1 \leq m \leq 10$ the order $\mathbf{Z}[\overline{T}]$ is the entire ring of integers or at least has only a small index. With the help the computer we obtain:

m	$H_1(X, \mathbf{Z})_{\text{tor}}$	upper bound	$\text{Vol}(X)$
1	4	13.54	2.3702...
2	2	9.58	1.0105...
3	2856582	8575220	177.8782...
4	32	122.47	22.1167...
5	5146	15731.73	111.5530...
6	288	1022.58	79.3724...
7	1288	4175.28	104.6757...
8	2	11.07	1.5189...
10	14	43.89	41.7309...

The values in the two right-hand side columns have been truncated. The particularly large values for $m = 3, 5$ are mostly caused by the fact that these number fields have exceptionally large Dirichlet regulators. Allow me to emphasize once more that the computation of the upper bounds requires the determination of the fundamental unit just as does finding the torsion group. Therefore, this estimate is truly not of any algorithmic use.

11. SMALLEST VOLUME

Firstly, we must ask: Is this question well-defined at all?

Usually, when one looks at questions like

$$(\text{complex surfaces}) \cap (\text{LCK manifolds})$$

as in Vaisman's paper [Vai87]⁴, or

$$(\text{real solvmanifolds}) \cap (\text{LCK manifolds})$$

as suggested in work of Hasegawa [Has05], we might primarily be interested in the existence of a Kähler or LCK metric at all. Once such exists, there can be many, at the very least we can rescale it ("Kähler cones"). In this sense the volume depends on choices and it is a pointless task to find a smallest volume among arbitrary choices. However, the situation is a little different for Oeljeklaus-Toma manifolds.

For finite volume hyperbolic n -manifolds X, X' (with $n \geq 3$) if there exists an isomorphism of fundamental groups $\pi_1(X, *) \xrightarrow{\sim} \pi_1(X', *)$, then there even exists an isometry

⁴This paper seems to have been written in response to Wall's study [Wal85], [Wal86]. Taking inspiration from Thurston's geometries, Wall asks which 4-dimensional geometries (= nice simply connected Riemannian real manifolds whose isometry group acts transitively and admits lattices) possess a complex structure so that the isometry action is holomorphic. He finds that a complex structure often exists, often unique, but not always Kähler.

$\phi : X \xrightarrow{\sim} X'$ (Mostow-Prasad Rigidity). In particular, the volume is a topological invariant; homeomorphic spaces must have the same volume. This makes it very interesting to study the possible volumes, and to search for a smallest volume.

For the Oeljeklaus-Toma manifolds $X(K; \mathcal{O}_K^{\times,+})$ the Proposition 1 creates a somewhat similar situation. We get a well-defined function

$$\text{Vol} : \left\{ \begin{array}{l} \text{spaces } X \text{ homeomorphic to an} \\ \text{Oeljeklaus-Toma manifold} \end{array} \right\} \longrightarrow \mathbf{R}$$

by associating to any X its “canonical model” $(\mathbf{H}^s \times \mathbf{C})/(\pi_1(X, *))$, which comes with the standard normalized Oeljeklaus-Toma metric. So at least after fixing once and for all a normalized metric (as we have done in this text), we get a well-defined volume and in particular well-defined infimum of volumes.

The situation might be quite different for the spaces $X(K; U)$ for $t > 1$ complex places. As Example 2 shows, there are different number fields K, K' and admissible subgroups U, U' so that there exists a diffeomorphism

$$\frac{\mathbf{H}^s \times \mathbf{C}^t}{\mathcal{O}_K \rtimes U} \xrightarrow{\sim} \frac{\mathbf{H}^s \times \mathbf{C}^t}{\mathcal{O}_{K'} \rtimes U'},$$

yet even if there happens to exist a normalized LCK metric (for example Battisti’s generalized Oeljeklaus-Toma metric, [Dub14, Appendix]), I would suspect the volumes to differ. Example 2 however says nothing about this since these spaces do not admit any LCK metrics for sure, as we explain *loc. cit.*

This being said and an overall normalization chosen, let us investigate whether there is a smallest volume. Certainly, the infimum of volumes could just be zero. For those readers who like the bridge to hyperbolic 3-manifolds as alluded to in §1, it should be said that there is a unique smallest compact orientable hyperbolic 3-manifold, the Weeks manifold [CFJR01], [GMM09]. Its volume is

$$\frac{3 \cdot 23^{\frac{3}{2}}}{4\pi^4} \zeta_K(2) \quad \text{for} \quad K := \mathbf{Q}[T]/(T^3 - T + 1).$$

This cubic number field K is the one whose discriminant has the smallest absolute value among all cubic fields. The volume of its Oeljeklaus-Toma manifold is ≈ 0.33714644 . Surprisingly, it turns out that this is also the smallest possible volume of an Oeljeklaus-Toma manifold with $s = 1$.

By quoting some rather hard results from analytic number theory and the geometry of numbers, one can show with little effort that, once fixing a number of real places s , the volume among all Oeljeklaus-Toma manifolds generally stays bounded away from zero:

Proposition 12. *For every $s \geq 1$ there exists a unique real number Vol_s so that the following holds:*

- (1) *All Oeljeklaus-Toma manifolds with fixed s have volume $\geq \text{Vol}_s$.*
- (2) *There exists at least one, but at most finitely many, actually attaining this minimal volume Vol_s .*
- (3) *We have the crude lower bound*

$$\text{Vol}_s \geq \pi \frac{(s+2)^{s+1}}{4^{s+2} \cdot 2^{s^2} \cdot s!}.$$

For the special case $s = 1$ there is a unique Oeljeklaus-Toma manifold of smallest volume, namely

$$\text{Vol}_1 = 0.337146 \dots$$

It is the one coming from the number field

$$K := \mathbf{Q}[T]/(T^3 - T + 1).$$

Proof. All the real work here lies in a deep result of Friedman [Fri89], based on earlier work of Remak and Zimmert. We have:

- For every number field K , apart from three exceptions with $[K : \mathbf{Q}] = 6$, we have $R_K > \frac{1}{4}$ ([Fri89, Theorem B]).
- For every number field K with $s = t = 1$ and $|\Delta_{K/\mathbf{Q}}| < 18.7^3$ we have $R_K/2 \geq 0.14$ ([Fri89, Prop. 2.2 and Table 2 for $(r_1, r_2) = (1, 1)$]).

If one is willing to accept far weaker bounds, a short proof of a lower bound for the regulator in terms of s can also be found in [Sko93]. Let X be an arbitrary Oeljeklaus-Toma manifold for a given $s \geq 1$. From the first estimate and Prop. 3 we readily obtain the bound

$$\text{Vol}(X) > \frac{(s+1)}{4^{s+1} \cdot 2^{s^2}} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|},$$

except for finitely many fields and we can ignore them as this does not affect the validity of our claim (since their regulators are explicitly known and listed in Friedman's work, we could also just work with the overall minimal regulator). Furthermore, there is the standard Minkowski discriminant estimate

$$\sqrt{|\Delta_{K/\mathbf{Q}}|} \geq \left(\frac{\pi}{4}\right) \frac{n^n}{n!} \quad \text{for } n := s+2$$

the degree of the field. Combining these inequalities, we arrive at

$$(11.1) \quad \text{Vol}(X) > \pi \frac{(s+1)(s+2)^{s+2}}{4^{s+2} \cdot 2^{s^2}(s+2)!}.$$

Work of Odlyzko, Martinet, and many others would give much better lower bounds for particular ranges of s , but this estimate suffices for our needs. Define

$$V_s = \{\text{Vol}(X) \mid X(K, \mathcal{O}_K^{\times,+}) \text{ for any } K \text{ with } t = 1 \text{ and given } s\} \subset \mathbf{R},$$

the set of all volumes that can occur for fixed s . This set is non-empty and bounded from below by Equation 11.1, so it will have some infimum $\wp := \inf(V_s)$. We now argue by contradiction: Suppose there is no X whose volume attains this infimum. This means that there exists a sequence of number fields K_n so that

$$(11.2) \quad \begin{aligned} \wp &= \lim_{n \rightarrow \infty} \text{Vol}(X(K_n, \mathcal{O}_{K_n}^{\times,+})) \\ &= \frac{(s+1)}{4^s \cdot 2^{s^2}} \cdot \lim_{n \rightarrow \infty} \sqrt{|\Delta_{K_n/\mathbf{Q}}|} \cdot R_{K_n}. \end{aligned}$$

The Hermite-Minkowski Theorem tells us that there are only finitely many number fields of bounded discriminant $|\Delta_{K/\mathbf{Q}}| < C$ for any $C \geq 0$, so if the sequence $(|\Delta_{K_n/\mathbf{Q}}|)_{n \geq 0}$ stays bounded, $\{K_0, K_1, K_2, \dots\}$ is actually a finite set and therefore some K_i will realize the infimum, contradicting our assumption. Thus, we must have $\lim_{n \rightarrow \infty} \sqrt{|\Delta_{K_n/\mathbf{Q}}|} = +\infty$. Hence, from Equation 11.2 we can deduce that $\lim_{n \rightarrow \infty} R_{K_n} = 0$. This contradicts Friedman's bound $R_{K_n} > \frac{1}{4}$. Thus, there exists at least one K_i with $\text{Vol}(X(K_n, \mathcal{O}_{K_n}^{\times,+})) = \wp$. If $\{K_i\}$ now denotes the (possibly infinite) set of all number fields realizing the volume \wp , that is

$$\wp = \frac{(s+1)}{4^s \cdot 2^{s^2}} \cdot \lim_{n \rightarrow \infty} \sqrt{|\Delta_{K_n/\mathbf{Q}}|} \cdot R_{K_n},$$

the same argument as above shows that the set $\{K_i\}$ must be finite, for otherwise the discriminants grow arbitrarily large, ultimately forcing regulators $\leq \frac{1}{4}$, which is impossible.

Next, consider the case $s = t = 1$: Firstly, (for $|\Delta_{K/\mathbf{Q}}| \geq 18.7^3$) the first Friedman estimate shows that

$$\text{Vol}(X) = \frac{1}{4} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \cdot R_K > \frac{1}{16} \cdot \sqrt{18.7^3} > 5.$$

Next, suppose $|\Delta_{K/\mathbf{Q}}| < 18.7^3$. Then the second Friedman estimate implies

$$\text{Vol}(X) = \frac{1}{4} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|} \cdot R_K > \frac{0.28}{4} \cdot \sqrt{|\Delta_{K/\mathbf{Q}}|}$$

and since the smallest possible discriminant of a cubic field is $|\Delta_{K/\mathbf{Q}}| = 23$, we deduce $\text{Vol}(X) > 0.335708$. Thus, we have a good lower bound for the smallest possible volume. Next, let us assume that K has a discriminant of absolute value larger than 23, so at least 24. Then Friedman's bound shows that

$$\text{Vol}(X) > \frac{0.28}{4} \cdot \sqrt{24} > 0.3429.$$

Since the Oeljeklaus-Toma manifold of $K := \mathbf{Q}[T]/(T^3 - T + 1)$ has the underlined volume in

$$0.335708 < \underline{0.3371\dots} < 0.3429,$$

we deduce that the minimal volume can (and is) attained only for number fields K with $s = t = 1$ and discriminant $|\Delta_{K/\mathbf{Q}}| = 23$. However, in the present case it is known that there exists only one number field with discriminant of absolute value 23. \square

Proof of Prop. 4. This is just a reformulation of the previous result, using that the dimension of an Oeljeklaus-Toma manifold is $\dim X = s + 2$. \square

After the Weeks manifold, the compact oriented arithmetic hyperbolic 3-manifold of next larger volume is the Meyerhoff manifold, [CFJR01]. It was shown by Chinburg [Chi87] to be arithmetic and to have volume

$$\frac{12 \cdot 283^{\frac{3}{2}}}{(2\pi)^6} \zeta_K(2) \quad \text{for} \quad K := \mathbf{Q}[T]/(T^4 - T - 1).$$

This quartic number field K has $s = 2$ and $t = 1$ real resp. complex places and discriminant -283 . It is known that the smallest discriminants for these numbers of places are as given on the left-hand side column in the following table:

$\Delta_{K/\mathbf{Q}}$	$\text{Vol}(X)$	min. polynomial
-275	0.0717	$T^4 - T^3 + 2T - 1$
-283	0.0745	$T^4 - T - 1$
-331	0.0921	$T^4 - T^3 + T^2 + T - 1$
-400	0.1196	$T^4 - T^2 - 1$
-475	0.1473	$T^4 - 2T^3 + T^2 - 2T + 1$

We leave it to the reader to show that the middle column indeed gives the smallest four possible volumes for Oeljeklaus-Toma manifolds with two real places. One can proceed as in the argument above, this time using Friedman's estimate $R_K/2 > 0.1835$ for $|\Delta_{K/\mathbf{Q}}| \leq 36^4$, [Fri89, Prop. 2.2 and Table 2 for $(r_1, r_2) = (2, 1)$].

Example 6. We will now determine the minimal volumes of Oeljeklaus-Toma manifolds for $s = 1, 2, 3, 4, 5$. We follow the same method as in the proof of Prop. 12, but suppress a number of details and just explain the general pattern. It would seem entirely hopeless to me to perform the necessary verifications below without the help of a computer. Using tables for minimal known discriminants, we first compile the following table:

$$(11.3) \quad \begin{array}{c|c|c|c|c|c} s & R_K^{\geq} & |\Delta_{K/\mathbf{Q}}|_{1^{st}} & |\Delta_{K/\mathbf{Q}}|_{2^{nd}} & \text{Vol of } 1^{st} & V_{2^{nd}}^{\geq} \\ \hline 1 & 0.28 & 23 & 31 & 0.33714 & 0.38974 \\ 2 & 0.367 & 275 & 283 & 0.07174 & 0.07235 \\ 3 & 0.6218 & 4511 & 4903 & 0.00515 & 0.00531 \\ 4 & 1.2376 & 92779 & 94363 & 0.0001146 & 0.0001133 \\ 5 & 2.7822 & 2306599 & 2369207 & 7.650 \cdot 10^{-7} & 7.478 \cdot 10^{-7} \end{array}$$

Here the column “ R_K^{\geq} ” lists a lower bound for the regulator of all number fields with given s and $t = 1$. We just copied these values from the work of Friedman ([Fri89, Table 2 for $(r_1, r_2) = (s, 1)$]), noting that his table spells out lower bounds for $R_K/2$. His values are only valid for discriminants smaller than certain bounds also given in [Fri89, Table 2], but these are harmless in all cases we deal with. Unsurprisingly so, as we are mostly interested in the smallest possible discriminants. The columns “ $|\Delta_{K/\mathbf{Q}}|_{1^{st}}$ ” and “ $|\Delta_{K/\mathbf{Q}}|_{2^{nd}}$ ” list the smallest and second smallest discriminant possible for the given s and $t = 1$. In principle there could be several number fields realizing the smallest discriminant, but in all cases we touch here, there is a unique one:

$$(11.4) \quad \begin{array}{c|c} s & \text{number field of } |\Delta_{K/\mathbf{Q}}|_{1^{st}} \\ \hline 1 & T^3 - T^2 + 1 \\ 2 & T^4 - T^3 + 2T - 1 \\ 3 & T^5 - T^3 - 2T^2 + 1 \\ 4 & T^6 - T^5 - 2T^4 + 3T^3 - T^2 - 2T + 1 \\ 5 & T^7 - 3T^5 - T^4 + T^3 + 3T^2 + T - 1 \end{array}$$

We compute their regulators with the help of a computer and therefore obtain the volumes of the Oeljeklaus-Toma manifolds associated to the number field of smallest possible discriminant. These values are listed in the column “Vol of 1^{st} ”. Next, we use Friedman’s bound to compute a lower bound on the volumes of all Oeljeklaus-Toma manifolds from number fields of discriminant at least the second smallest, i.e. in the column “ $V_{2^{nd}}^{\geq}$ ” we list

$$(11.5) \quad \frac{(s+1)}{4^s \cdot 2^{s^2}} \sqrt{|\Delta_{K/\mathbf{Q}}|_{2^{nd}}} \cdot (\text{Friedman bound } R_K^{\geq}).$$

The cases $s = 1, 2, 3$ are obvious now: We find that as soon as we use number fields whose discriminants are second smallest or larger, we will exceed the volume of the Oeljeklaus-Toma manifold made from the number field of smallest discriminant. The case $s = 4$ is more involved since we see that the Oeljeklaus-Toma manifold of the unique sextic field of smallest discriminant has a volume strictly larger than a volume that could hypothetically occur for the second smallest discriminant as well. In fact, the second smallest discriminant for $s = 4$, that is -94363 , is realized by the number field of

$$T^6 - 2T^4 - 2T^3 + 3T + 1.$$

We compute its volume to be 0.000116, so it is not smaller. The next larger discriminant is known to be $|\Delta_{K/\mathbf{Q}}|_{3^{rd}} = 103243$, and Friedman’s bound as in Equation 11.5 yields a minimal volume of 0.00011851 for discriminants $\geq |\Delta_{K/\mathbf{Q}}|_{3^{rd}}$. This settles the case: Still,

the manifold coming from the number field of smallest discriminant has also the smallest volume. For $s = 5$ the same happens. The second smallest discriminant is realized by

$$(11.6) \quad T^7 - 4T^5 + 3T^3 - T^2 + T + 1$$

and we compute its volume to be $7.88 \cdot 10^{-7}$, so its volume is larger. We have $|\triangle_{K/\mathbf{Q}}|_{3rd} = 2616839$ and Friedman's bound shows that for this and larger discriminants the volumes must be at least $7.85 \cdot 10^{-7}$. This confirms that we have found the smallest one, but do not know for sure whether Equation 11.6 defines the second smallest one.

We conclude: The values listed under “Vol of 1st” in Table 11.3 provide the smallest possible volume for the given s , and in each case are realized by only a single manifold; and these are the ones listed in Table 11.4.

Although all of the above computations might suggest that the volumes follow the ordering of ascending discriminants, this simple pattern completely collapses as we get farther from the minimal volumes. The polynomials in the following table have been chosen rather at random, but ordering the rows by increasing volume shows that this does not imply much about the ordering of the discriminants:

$\triangle_{K/\mathbf{Q}}$	Vol (X)	min. polynomial
−1931	0.7162	$T^4 + 3T + 1$
−6371	3.0870	$T^4 + 13T + 1$
−8123	3.5939	$T^4 - 4T^3 - T - 1$
−12675	4.6792	$T^4 - 8T^3 - T - 1$
−6656	5.3600	$T^4 - 4T + 1$
−16619	7.5061	$T^4 - 5T + 1$
−8684	9.2152	$T^4 - 6T + 1$

Although Friedman's estimates are very non-trivial results, the result that the Weeks manifold has smallest volume among compact hyperbolic 3-manifolds is of a completely different level of complexity. This remark truly applies to any comparison we make between Oeljeklaus-Toma manifolds and hyperbolic or product-hyperbolic geometries in this text.

Acknowledgement. *I would like to express my sincere gratitude to Victor Vuletescu for teaching me a lot of things, not all of them of mathematical nature. This note is a direct result of his inspiring ideas about the interplay of geometric and arithmetic conditions in Oeljeklaus-Toma manifolds. I also thank Chris Wuthrich for introducing me to SAGE.*

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12. APPENDIX

Computer Code 1. The computations underlying Example 2 can be confirmed in an automated fashion by computer algebra systems. The following code is written for SAGE [S⁺], largely using PARI/GP [The14]. Firstly, we confirm that S was a generator of the group of units (up to torsion):

```
L1.<s> = NumberField(x^3+x+1)
print L1.unit_group().gens()
```

This computation can also be done by hand using the Minkowski bounds; but remember that this verification was actually not needed for the validity of the example. Next, we check the crucial fact that the order J_i is maximal, i.e. that it is the ring of integers:

```
from sage.rings.number_field.order import *
E.<s,t> = NumberField([x^3+x+1,x^3-x+2])
C.<w> = E.absolute_field()
V, from_v, to_v = C.vector_space()
J = span([to_v(1), to_v(s), to_v(s^2), to_v(t), to_v(t^2),
to_v(s*t), to_v(s^2*t), to_v(s*t^2), to_v(s^2*t^2)],ZZ)
O = AbsoluteOrder(C, J)
print O.is_maximal()
```

Adapt the minimal polynomial for t to check both cases $i = 2, 3$. Finally, we check that the compositum of the Galois closures has degree 216:

```
L1.<s> = NumberField(x^3+x+1)
L2.<t> = NumberField(x^3-x+2)
L3.<u> = NumberField(x^3-x+1)
H1 = L1.galois_group(names='b').splitting_field()
H2 = L2.galois_group(names='c').splitting_field()
H3 = L3.galois_group(names='d').splitting_field()
B = H1.composite_fields(H2)[0]
C = B.composite_fields(H3)[0]
print C.degree()
```

Of course it would not be particularly hard to perform this computation by hand, just a bit tedious.

Computer Code 2. We discuss the determination of the ideal $J(U)$, Definition 1, by computer. We have used this for our Example 1. The following code runs through the number fields generated by the minimal polynomials $Z^3 - Z + h$, whenever these are irreducible, for $h = 1, \dots, 9$. In this particular case these number fields have $s = t = 1$ real resp. complex places, so $\mathcal{O}_K^\times \simeq \langle -1 \rangle \times \mathbf{Z} \langle u \rangle$, where u is a fundamental unit. For these minimal polynomials the single real embedding of the fundamental unit always happens to be negative. This follows from Descartes' Rule of Signs: The polynomial rewritten in $-Z$ is $-Z^3 + Z + h$, which has precisely one sign change among its coefficients. Therefore, it must have a single negative real root. Hence, $\mathcal{O}_K^{\times,+} \simeq \mathbf{Z} \langle -u \rangle$ and the ideal $J(\mathcal{O}_K^{\times,+})$ is generated by the single element $1 - (-u) = 1 + u$ by Lemma 2.

```
z = QQ['z'].0
for h in range(1,10):
    if (z^3-z+h).is_irreducible():
```

```

L.<s> = NumberField(z^3-z+h)
U = L.unit_group()
T = 1+U.gen(1)
J = L.ideal(T);
print s.minpoly(), " -> ", factor(J.norm())

```

Note that SAGE always returns the unit group in the format so that `U.gen(0)` is the torsion generator and `U.gen(1)` the non-torsion generator. Hence, in this particular case the ideal J needs to be generated by `1+U.gen(1)`. This code can easily be adapted to similar computations. For example, for the polynomials $Z^7 - Z - h$ we will have $s = 1$ and $t = 3$ real resp. complex places. Any such polynomial has exactly one sign change in its coefficients, so by Descartes' Rule it has precisely one positive real root. Hence, $\mathcal{O}_K^\times = \{\pm 1\} \times \mathcal{O}_K^{\times,+}$ and therefore $J(\mathcal{O}_K^{\times,+})$ is generated by the elements $1 - u$, where u runs through the generators of $\mathcal{O}_K^{\times,+}$, again by Lemma 2. Replace the definition of `T` by

```
T = [(1-U.gen(i)) for i in range(1,len(U.gens()))],
```

since we now will have several generators. We discard the generator at `i=0` since this is again the torsion generator.